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Estimation of the dynamic characteristics of physical systems with the aid of a Fourier synthesizer

Balfe, John M.

Massachusetts Institute of Technology

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John M. Balfe

and

David J. Woodard

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by

John M. Balfe, Lieutenant, U. S. Navy
B.S., U.S. Naval Academy, 1946
B.S., U.S. Naval Postgraduate School, 1955

David J. Woodard, Lieutenant, U.S. Navy
B.S., U.S. Naval Academy, 1949
B.S., U.S. Naval Postgraduate School, 1955

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by

John M. Balfe

David J. Woodard

Submitted to the Department of Aeronautical Engineering on May 21, 1956, in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

During the design stage of automatic control systems for aircraft, fire control or other physical systems, system components must be selected. Values of the parameters of these components must be determined so as to produce the desired dynamic properties of the resultant system.

The method of estimating dynamic characteristics employed in this investigation is based on the encirclement criteria of Cauchy's Residue Theorem. A computer was developed to aid in this estimation by making certain modifications to the data presentation of the Fourier Harmonic Synthesizer.

This investigation indicated that dynamic characteristics may be readily estimated by the method to be described for any selected set of system parameters. System parameter specifications and limits may also be set when the dynamic characteristics are specified.

Several examples are included to illustrate the techniques involved and to indicate the suitability of applying this method to aid in the analysis of higher order systems.

Thesis Supervisor:	Sidney Lees
Title:	Assistant Professor Aeronautical Engineering

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The graduate work for which this thesis is a partial requirement was performed while the authors were assigned to the U. S. Naval Administrative Unit, Massachusetts Institute of Technology, Cambridge, Massachusetts.

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OBJECT

The object of this investigation is to present suitable techniques, based on the Cauchy Residue Theorem and utilizing a modified Fourier Harmonic Synthesizer as a computer, to aid in the estimation of the dynamic characteristics of physical systems.

CHAPTER 1

INTRODUCTION

In the design of physical systems some of the most important considerations are the dynamic properties of the resultant system. It will be shown in Summary 1 that the differential equation describing the system may be written with the input terms on the right side. By setting the right side equal to zero the homogeneous equation is obtained. This equation determines the transient modes of the system, which are independent of the input. The actual input affects the magnitude of the transient modes but has no effect on the damping ratios, the oscillation frequencies, or response times of the transients. These characteristics are determined by the component parameters. Once the system is built, control over these dynamic properties is restricted. Since these properties have a prominent effect on the performance of the system, the engineer must have some means of predicting them during the design stage.

Exact determination of system parameters is often neither necessary nor desirable in the early stages of design if limiting values of the parameters can be estimated in terms of the dynamic characteristics of the system. Such an estimate will enable the engineer to set limits on the component specifications in the initial stages of design.

The method of estimation of system dynamic characteristics described in this thesis is based on a concept first advanced by Sidney Lees in Massachusetts Institute of Technology Instrumentation Laboratory Report R-71 of April 1954 [1].

The Mapping Theorem, described in Instrument Engineering Volume II [2] and in Theory of ServoMechanisms, [3] and the Cauchy Residue Theorem described by Guillemin [4] and others, are the basis for this concept.

Lees applies this theorem [1] and describes a procedure by which dynamic characteristics may be estimated from polynomial contours approximated by the computation of certain selected points.

If these contours can be accurately computed and displayed by some device, the application of the Cauchy Theorem to this problem should be greatly facilitated.

A Fourier Harmonic Synthesizer built at the Massachusetts Institute of Technology Instrumentation Laboratory and described by Instrumentation Laboratory Engineering Memo 6445-E-42 [5] was used for this purpose. Modifications to the existing installation and the techniques employed to utilize this device as an aid in the estimation of the dynamic characteristics of physical systems are described in the succeeding chapters.

CHAPTER 2

MATHEMATICAL BACKGROUND

The performance of physical systems may often be described by linear differential equations with coefficients that are constant once they have been decided upon. The properties of the components of the system may be obtained from the equation coefficients.

The complete solution of the linear differential equation consists of the transient solution, plus the forced solution as is shown in Summary 1.

The transient solution of the linear differential equation is the solution of the homogeneous equation, which is obtained by setting the right hand side of the differential equation equal to zero.

SUMMARY 1

As expressed in Instrument Engineering Volume II [2], a typical ordinary linear differential equation has the form

$$\sum_{i=0}^{n} a_i \frac{d^i V}{dt^i} = \sum_{k=0}^{m} b_k \frac{d^k u}{dt^k} \qquad (1)$$

The homogeneous equation is

$$\sum_{i=0}^{n} a_i \frac{d^i V}{dt^i} = 0 \qquad (2)$$

The associated characteristic equation is

$$\sum_{i=0}^{n} a_i z^i = 0 \qquad (3)$$

A fundamental theorem of algebra described in Summary 24-1 Instrument Engineering Vol. II [2] states that every equation of nth degree has n and only n roots. The roots of Equation (3) are $z_1, z_2, z_3, \ldots z_n$. In general the roots are complex numbers so that

$$z_i = x_i + jy_i \tag{4}$$

When all roots are distinct, transient solutions of Equation (1) have the form

$$V_{(tr)} = \sum_{i=1}^{n} c_i e^{z_i t} = \sum_{i=1}^{n} c_i e^{x_i t} e^{jy_i t} \tag{5}$$

Roots of the characteristic equation must have the dimensions of inverse time (from Equation (5)). Real roots may be expressed as

$$z_i = x_i = - \frac{1}{(CT)_i} = - \frac{1}{\tau_i} \tag{6}$$

where $(CT)_i = \tau_i$ = characteristic time of real root.

Complex roots may have the form

$$z_k = x_k + jy_k = - \zeta_k \omega_{n_k} + j \omega_{n_k} \sqrt{1 - \zeta_k^2} \tag{7}$$

where $\zeta_k = (DR)_k$ = damping ratio of kth root,

ω_{n_k} = undamped natural frequency of kth root, and $x_k = -\zeta_k \omega_{n_k} = -\frac{1}{(CT)_k} = -\frac{1}{\tau_k}$ where $(CT)_k \triangleq \tau_k =$ characteristic time of k<u>th</u> root.

In polar coordinates, the k<u>th</u> complex root form is

$$z_k = x_k + jy_k = \omega_{n_k} e^{j\theta} = \omega_{n_k} e^{j\left[\pi - (ADR)_k\right]} =$$

$$\omega_{n_k} e^{j(\pi - A_{\zeta_k})} \tag{8}$$

where $\theta = \pi - (ADR)_k = \pi - A_{\zeta_k}$

and $(ADR)_k = A_{\zeta_k} = \cos^{-1} \zeta_k$ = angle of damping ratio of the k<u>th</u> root.

The transient solution determines what oscillatory modes can be present, and their mode shapes, and also determines the extent to which the dynamic performance of the system will be affected by the specified input.

The forced solution reflects the effect of the input and determines the relative magnitude of the modes indicated in the transient solution superimposed on the forced solution. It also determines the system response time due to the effect of the input on the mode shape but has no effect on the damping or oscillation frequencies of the second order modes. The response time for the system will, in no case, be greater than that indicated by the slowest decaying term of the transient solution.

This investigation is not restricted to any specified input and considers only dynamic effects resulting from the roots of the characteristic equation. The analysis of this part of the complete performance equation gives information on dynamic properties which greatly affect the performance of the system for any arbitrary input and it is to this end that the investigation is directed.

The roots of the homogeneous equation expressed as a characteristic equation with the variable z are defined in M. I. T. Instrumentation Laboratory Report R-71 [1] as the dynamic characteristics of the associated system. The dynamic characteristics can be determined if the roots of the characteristic equation can be found by factoring or other means.

Lees [1] discusses the important considerations in the investigation of transient solutions of most practical systems. One consideration is that the transient solutions decay to zero, thus providing absolute stability. In addition, it is necessary that the transient solutions decay rapidly enough to meet specified conditions; this is the relative transient stability requirement. The damping ratio and the oscillation frequency are also important considerations in investigating the transient solutions. These are the dynamic characteristics and are apparent from the transient solution or roots of the characteristic equation. In this thesis, the terms τ, ζ and ω_n specified by the roots of the characteristic equation are considered to be the dynamic characteristics of the physical system. The equivalence of these terms and the characteristic equation roots are shown in Figure 1.

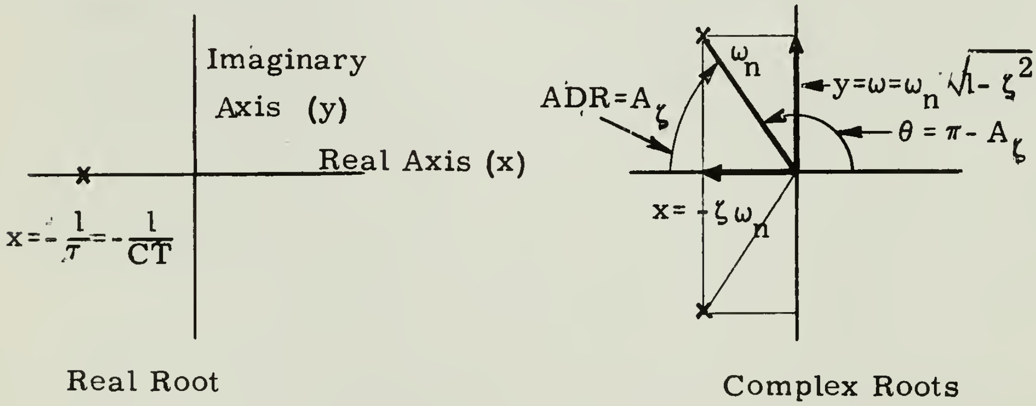


Figure 1: Roots of the Characteristic Equation as Represented on the Complex Plane.

Absolute transient stability of the system exists when all of the roots of the characteristic equation have negative real parts, i.e., when x_i and x_k are negative, $e^{x_i t}$ and $e^{x_k t}$ decay to zero as $t \rightarrow \infty$.

Relative transient stability exists when the smallest of the magnitudes of the real parts of the roots is greater than a specified reference magnitude, i.e., if $|x_{i_{mp}}| > |x_{ref}|$ then

$(CT)_i < (CT)_{ref}$ ($x_{i_{mp}}$ is most positive real part of all the roots).

This has been discussed in greater detail in Instrument Engineering, Volume II [(2)], Chapter 20.

The damping ratio and oscillation frequency of the system are related to the real and imaginary values of the complex roots as shown in Equation (7).

In many practical systems the dynamic characteristics are restricted to a certain desired sector or range depending on the relative transient stability requirements and specified undamped natural frequency and damping ratio limitations. These sectors and ranges are shown in Figure 2 below. Specification of a sector imposes limitations on the roots and thus the coefficients of the characteristic equation, and it is desired to determine the range of each coefficient which will provide dynamic characteristics within specified limits. Figure 2 shows how these limitations may be investigated with respect to the characteristic equation complex plane root loci. These contours are useful when investigating for relative stability, oscillation frequency and damping ratio. In most practical systems the specifications impose limitations on two or all three of these characteristics and a contour similar to Figure 2-d is used.

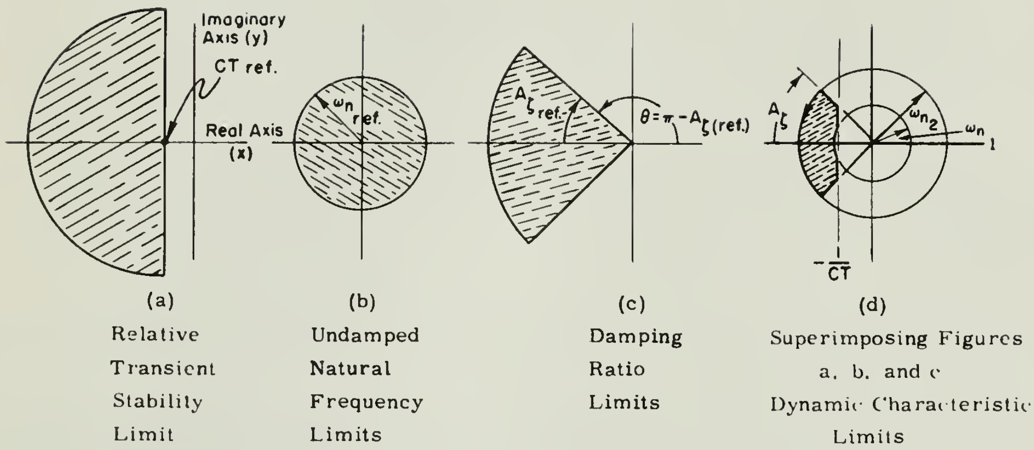


Figure 2: Standard Contours

This investigation is concerned with the techniques concerning the solution of two problems: determination or estimation of the dynamic characteristics, and fixing the limitations and relationships of the coefficients. Fixation of the coefficients of the characteristic equation determines the limits of the component parameters in the actual system.

The simpler case, that of locating the roots of the characteristic equation with known constant coefficients will be taken up first. This case will be treated in some detail since all the principles involved are equally applicable in estimating the dynamic characteristics from equations with adjustable coefficients.

In this presentation, the methods used in estimating the dynamic characteristics of physical systems are based on Cauchy's Residue Theorem, as presented by Guillemin [4] and others.

From Equation (3),

$$\sum_{i=0}^{n} a_i \; z^i = 0$$

This is expanded as

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_o = 0 \qquad (9)$$

From Cauchy's Residue Theorem, if $P(z)$ is a polynomial of z, $P'(z)$ is the derivative of the polynomial with respect to z, and C is a closed countour on the z complex plane containing N roots of $P(z) = 0$,

$$\oint_C \frac{P'(z)}{P(z)} dz = j\ 2\pi\ N \qquad (10)$$

This integral may be interpreted to mean that the argument of $P(z)$ increases exactly $2\pi N$ times as the closed contour C is traced out in a positive sense, or that the $P(z)$-plane path encircles its origin N times. This is also referred to as the mapping theorem, in Instrument Engineering, Volume II [(2)], and is illustrated in Figure 3.

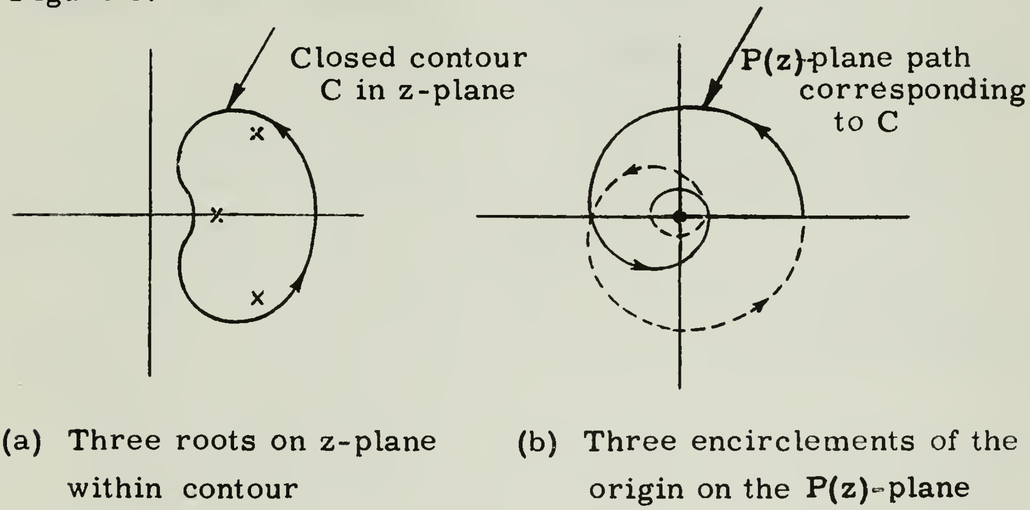


Figure 3: Illustration of Mapping Theorem

A closed contour similar to Figure 2-d and corresponding to Figure 3-a may be drawn on the characteristic equation root complex plane, in which it is desired to investigate for roots of the characteristic equation. By substituting values of the variable lying along the complex plane contour of Figure 2-d, a corresponding contour on the P (z) plane corresponding to Figure 3-b may be determined. The number of encirclements of the P(z) plane origin is the number of polynomial roots within the root complex plane contour. As will be shown in the examples, the polynomial P(z) of Equation (9) usually includes a constant term a_o. It is convenient to write the polynomial as $P(z)-a_o$ and $-a_o$ becomes the P(z) plane encirclement point.

The P(z) plane path of Figure 3-b may be calculated by hand by substituting values of the variable along the contour of Figure 2-d. This becomes impractical since a large number of calculations is required if a complete and accurate P(z) plane path corresponding to the z plane contour of Figure 2-d is desired. Chapter 3 describes how the Fourier Synthesizer may be used to obtain the information provided by the P(z) plane path.

CHAPTER 3

APPLICATIONS OF THE FOURIER HARMONIC SYNTHESIZER

The Fourier Harmonic Synthesizer may be used to facilitate the application of Cauchy's Residue Theorem to this problem, reduce the required calculations, increase the accuracy and bring other related items of interest to light.

It has been shown that the polynomial in z may be written in the form

$$P(z) - a_o = a_n R^n e^{jn\theta} + a_{n-1} R^{n-1} e^{j(n-1)\theta} + \dots$$

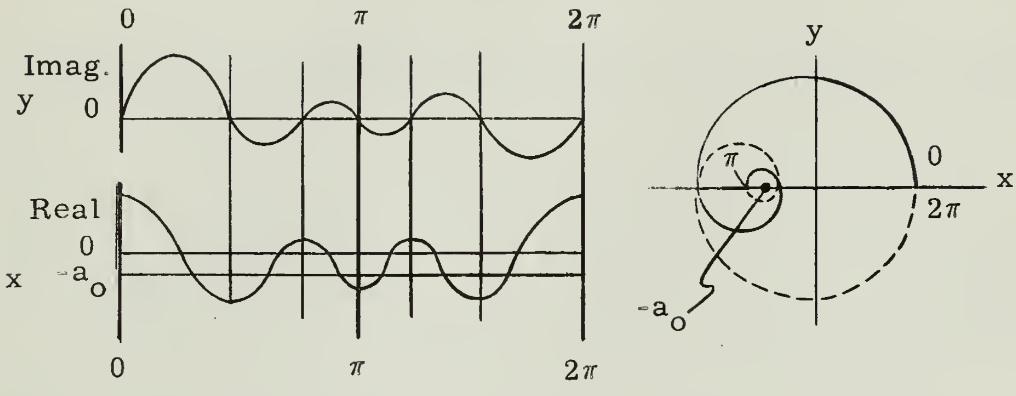
$$\dots + a_1 R e^{j\theta} \qquad (11)$$

by setting $z = Re^{j\theta}$.

This polar form of the equation is suitable for setting into the harmonic synthesizer. The method of scaling to obtain full scale deflection of the synthesizer and maximum accuracy thereby is given and explained in Example 1.

Each of the coefficients (R^n, R^{n-1}) corresponds to the radius of the contribution of the particular term of each order as the variation is traced out from $\theta = 0$ to $\theta = 2\pi$.

The synthesizer traces the polynomial $[P(z)]$ plane from $\theta = 0$ to $\theta = 2\pi$ for the selected R. The plot as recorded on the Sanborn Recorder is displayed as separate imaginary (y) and real (x) components. The polynomial contour (P(z) plane) which may be constructed from the x and y components, is symmetrical about the real axis. The real component is a mirror image about 180^o and the imaginary component crosses through zero at 180^o with the 180^o to 360^o portion being an inverted mirror image. Therefore, as is indicated in Figure 4, only the 0° to 180° portion is required.



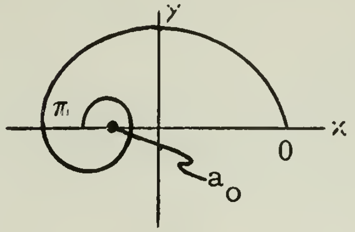
(a) Synthesizer Component Traces

(b) Polynomial Contour Represented by Synthesizer Traces

Figure 4

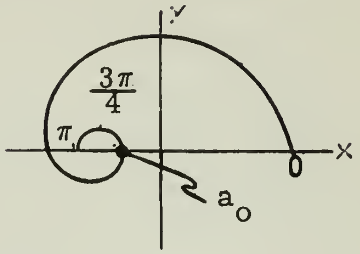
The P(z) plane may be drawn from the imaginary and real components, and the number of encirclements of the constant term counted. This, as Cauchy's Residue Theorem states, gives the number of roots that may be found at smaller radii (R) than that selected.

The major item of interest is the y (imaginary) cross-over (where the y component equals zero) in relation to the constant term a_o on the negative real axis. This point is represented by a horizontal line at $x = -a_o$ on the x (real) component plot. Only the θ at y cross-overs need be noted from the imaginary component plot, and from the real component plot only the fact of whether the corresponding points are more negative or less negative than the constant term need be noted in order to sketch rapidly and adequately the approximate P(z) plane. If at a y (imaginary) cross-over, the x (real) component crosses the constant term line at that point, roots occur at that radius, one real root if $\theta = \pi$, two complex conjugate roots if θ equals anything other than π. This immediately gives the (ADR) and (DR) of those roots, since (ADR) = $\pi - \theta$ and (DR) = cos $(\pi - \theta)$, as well as the magnitude of the roots, since $R = \omega_n$. The typical P (z) planes of Figure 5 will illustrate this.



Three encirclements of a_o
3 roots at $\omega_n < 1$

(a) P(z) contour for R= 1



One encirclement of a_o
1 root at $\omega_n < .8$

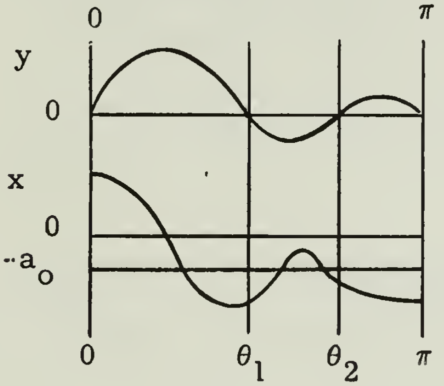
2 complex roots at $\omega_n = .8$

$\theta = \frac{3\pi}{4}$, $\zeta = \cos \frac{\pi}{4} = .707$

(b) P(z) contour for R = .8

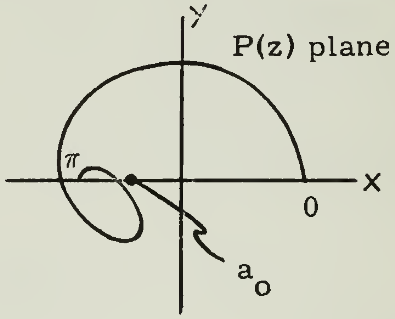
Figure 5: Typical P(z) Contours

The approximate P(z) plane need not be sketched at all since the encirclement criteria is immediately apparent from the x and y component traces directly. If during the interval between two y cross - overs, the real plot crosses the constant line, and remains on the opposite side when the y plot again reaches zero, one encirclement is indicated. That is, one root on the complex plane exists at a radius less than that for which the plot was made. This is illustrated in Figure 6.



One encirclement between 0 and θ_1
No encirclements between θ_1 and θ_2 or π

(a) One Encirclement Indicated by Synthesizer Traces



(b) One Encirclement Indicated by P(z) Contour

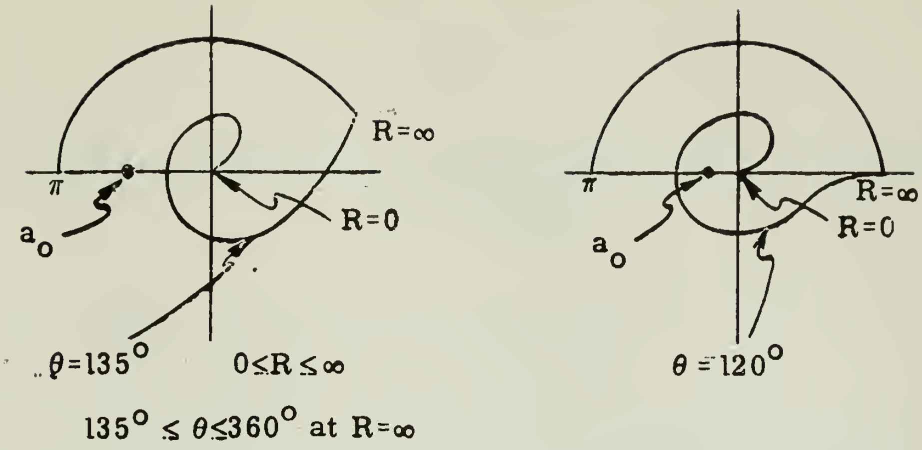
Figure 6

It should be noted that from the component or P (z) plot of just a single radius, no conclusion should be made as to whether the roots indicated are real or complex, nor estimate the (DR) of such roots unless some additional information is known about the equation being tested or unless the x plot crosses the constant line at or very near the θ at which the y cross-over occurs. Caution should be exercised when attempting to draw concrete conclusions in estimating (DR) from plots of x plots crossing the constant line near a y cross-over until the effects of R variation for the particular equation is fully appreciated and relative magnitude of synthesizer indications and tape scaling factors are considered. Examples (1) and (2) demonstrate, among other things, how an error in estimating (DR) may occur by considering only the results of a single run.

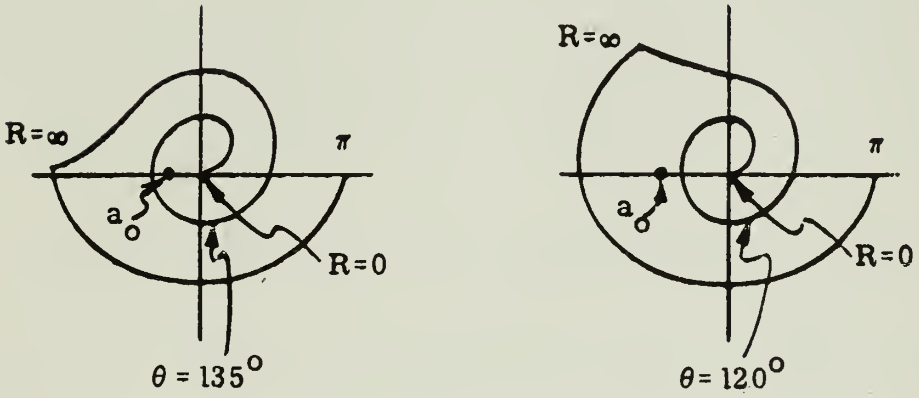
By making several runs of constant R for varying θ on the machine, a composite plot may be constructed as is shown in Example (3).

The magnitude of the roots may be estimated within the limits of the selected R's by the criteria described above. Curves connecting points of the various R plots for equal angles (θ) produce curves that are actually constant θ curves for varying R. The number of encirclements of the constant term by these constant θ curves give the number of roots at θ's greater than the value of θ for that curve. This gives the limits on the damping ratio, since $\zeta = \cos \ (180^{\circ} - \theta)$.

As is indicated in Figure 7, the number of encirclements of the constant θ curves is obtained by considering θ to be held constant until an infinite radius (R) is reached then permitting θ to complete its sweep to 180°. The latter part of the sweep (@ infinite R) will never appear on any trace or plot but must be considered in order to determine encirlements.



(a) Constant θ curves for
Third Order Equation



(b) Constant θ curves for
Fourth Order Equation

Figure 7: Typical Constant θ - Variable R curves

CHAPTER 4

TECHNIQUES INVOLVED IN THE TREATMENT OF SPECIFIC EXAMPLES

EXAMPLE 1: Constant Coefficient Case

In order to scale the problem in a manner suitable for setting into the synthesizer, the polynomial in z is written in the polar form as shown below.

The polar form coefficients are determined for each of the trial radii (R's) selected. The scale setting is then determined so that the largest coefficient may correspond to the largest synthesizer scale setting (1. 0) in order that the maximum synthesizer accuracy may be obtained. This simple set of slide rule ratios lends itself to tabular presentation as is indicated in Table I, page 38 which follows the examples.

$$P(z) - 52 = z^5 + 13 z^4 + 60 z^3 + 120 z^2 + 124 z$$

$$= R^5 e^{j5\theta} + 13 R^4 e^{j4\theta} + 60 R^3 e^{j3\theta}$$

$$+ 120 R^2 e^{j2\theta} + 124 Re^{j\theta}$$

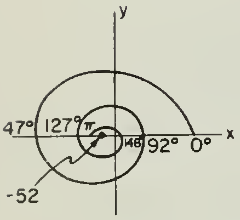
since $z = Re^{j\theta}$

EXAMPLE 1

Constant Coefficient Case

The Sanborn record of the synthesizer component traces are sketched and summarized below for convenience. The values at the important points and cross-overs are included and the corresponding P(z) plots sketched in order that the effect of R variation may be readily observed. The conclusions and estimates based on the individual runs are given with the corresponding traces in order that the technique used in the investigation may be followed conveniently.

$$P(z) - 52 = z^5 + 13\,z^4 + 60\,z^3 + 120\,z^2 + 124\,z$$

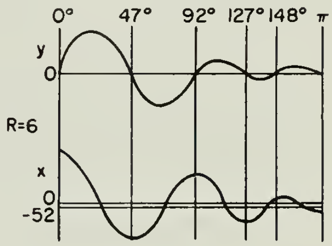


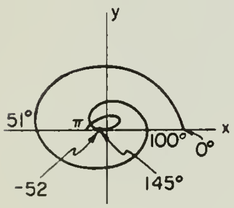
5 encirclements; hence 5 roots at R < 6
At least 1 real root.
One complex pair may be expected with a θ at

$148^\circ > \theta > 127^\circ$

or

$127^\circ > \theta > 92^\circ$





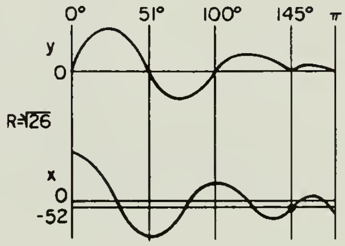
2 roots @ $R = \sqrt{26}$

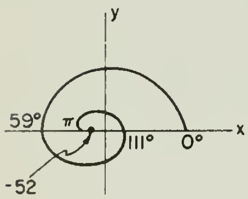
$\theta = 145^\circ \quad A_\zeta = 35^\circ$

(The run for R=6 indicated the possibility of roots @ $\theta = 145^\circ$. The other indication, i.e. $127^\circ > \theta > 92^\circ$ did not materialize)

3 encirclements
3 roots @ $R < \sqrt{26}$

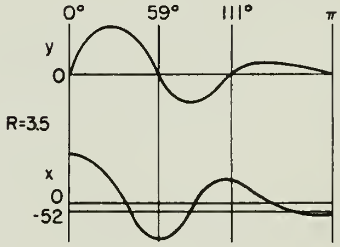
Due to the position of the loop in the P(z) plot the only safe estimate is that if there is another complex pair, it should occur at $100^\circ < \theta < \pi$

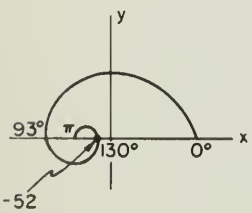




3 roots @ R < 3.5

If there is another complex pair at R < 3.5 it is more likely to occur between 111° and π then between 59° and 111°. (The basis of this estimate is the position of the constant term and observing the behavior for many other equations)

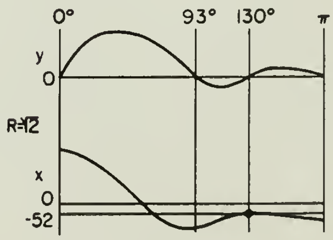




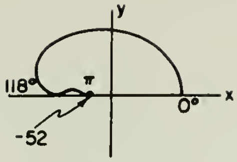
2 roots @ $R = \sqrt{2}$

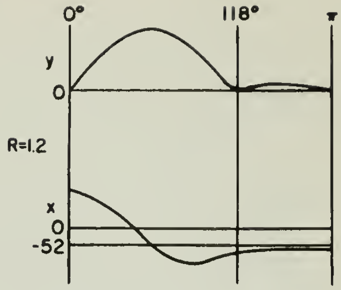
$\theta = 130^\circ \quad A_\zeta = 50^\circ$

1 root (real) @ $R < \sqrt{2}$



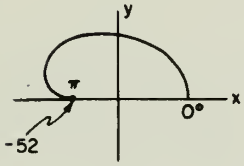
Example 1 (cont.)

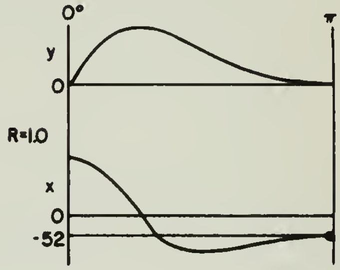




1 root @ R<1.2

(The polynomial contour plot and component traces indicate the real root should be very close to R = 1.2)

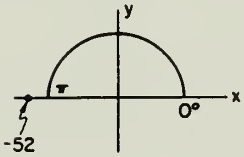


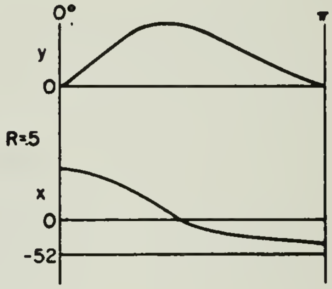


1 root @ R=1

$\theta = \pi \quad A_\zeta = 0$

All roots have been located





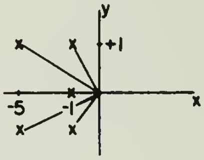
No roots occur at R<.5

This checks with previous runs since all roots have been located at R's > .5

The equation $\quad P(z) = z^5 + 13z^4 + 60z^3 + 120z^2 + 124z + 52$

has roots of $\quad z = -5 \pm j \quad z = -1 \pm j \quad z = -1$

$(\omega_n = \sqrt{26},\ A_\zeta = 11^\circ)\ (\omega_n = \sqrt{2}, A_\zeta = 45^\circ)\ (\alpha = 1, A_\zeta = 0)$



The actual values of the roots of the equation agree very well with those indicated by the synthesizer.

EXAMPLE 2

Constant Coefficient Case

Example 2 is of a form very similar to Example 1, even to the extent of an increasing magnitude of the coefficients with decreasing powers of the variable.

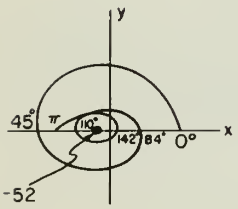
Example 1: $P(z) = z^5 + 13z^4 + 60z^3 + 120z^2 + 124z + 52$

Example 2: $P(z) = z^5 + 5z^4 + 36z^3 + 88z^2 + 108z + 52$

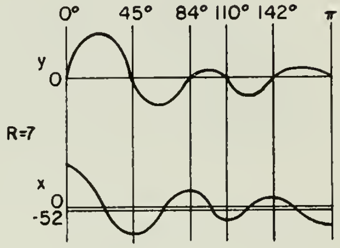
(The constant term (a_o = 52 in both examples) while greatly affecting the roots has no effect on the shape or magnitude of the component traces or polynomial contour plot)

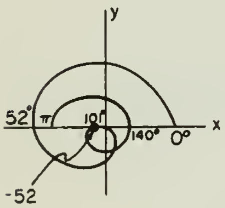
These two examples, while similar in form, show quite different behavior and indicate that an estimate of θ (or A_ζ) based on a single run may be considerably in error.

$$P(z) - 52 = z^5 + 5z^4 + 36z^3 + 88z^2 + 108z$$



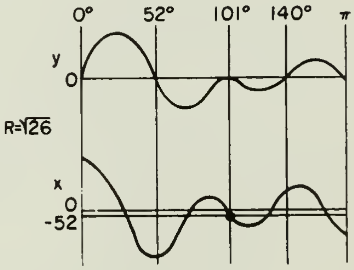
5 encirclements, 5 roots @ R < 7
If complex roots exist, one pair may be expected @ $142^\circ > \theta > 110^\circ$
or $110^\circ > \theta > 84^\circ$

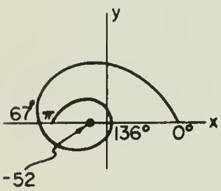




2 roots @ $R = \sqrt{26}$
$\theta = 101^\circ$ $A_\zeta = 79^\circ$
Second estimate from the R=7 run is substantiated. (Compare the $R = \sqrt{26}$ runs of Example 1 and Example 2)

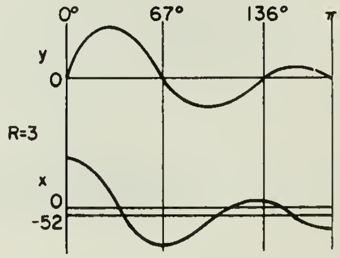
3 roots @ $R < \sqrt{26}$

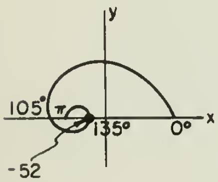




3 roots @ R < 3

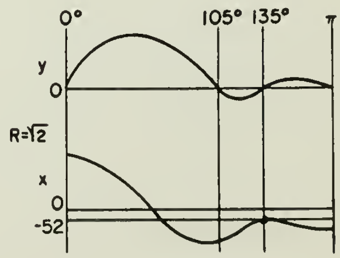
If more complex roots exist, it cannot be stated positively whether they will be at $67^\circ < \theta < 136^\circ$
or $136^\circ < \theta < \pi$
The relative magnitude of the trace peaks indicate however, that θ will be closer to 136° than either 67° or π



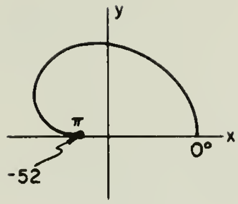


2 roots @ $R = \sqrt{2}$
$\theta = 135^\circ$ $A_\zeta = 45^\circ$

1 root (real) @ $R < \sqrt{2}$



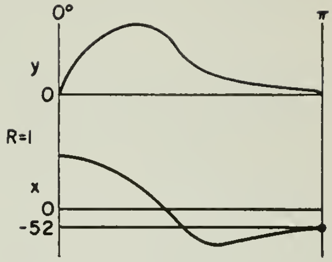
Example 2 (cont.)

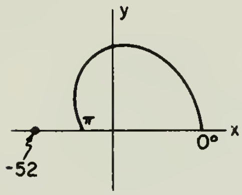


1 root @ R=1

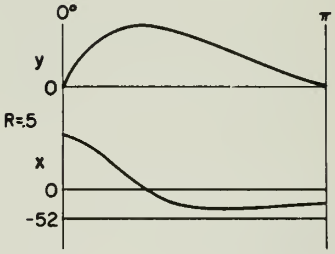
$\theta = \pi \quad A_\zeta = 0$

All 5 roots have been located.
No indication of roots should be found at R's < 1





No roots @ R<.5

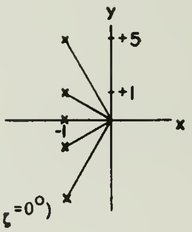


The polonomial:

$$P(z) = z^5 + 5z^4 + 36z^3 + 88z^2 + 108z + 52$$

has roots: $z = -1 \pm 5j \qquad z = -1 \pm j \qquad z = -1$

$(\omega_n = \sqrt{26}\ ,\ A_\zeta = 79^\circ) \quad (\omega_n = \sqrt{2}\ ,\ A_\zeta = 45^\circ) \quad (\alpha = 1, A_\zeta = 0^\circ)$

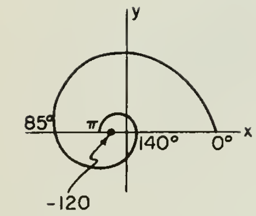


In this example the actual values of the roots agree almost exactly with the values indicated by the synthesizer.

EXAMPLE 3

Constant Coefficient Case

$$P(z) - 120 = z^3 + 10z^2 + 44z$$

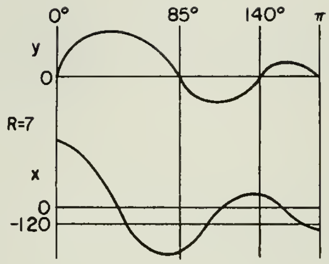


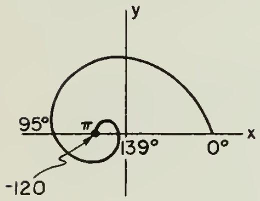
3 encirclements

3 roots @ R<7

(Possibility of one real root is indicated at R less than but close to 7)

(Suspect one complex pair @ $85^\circ < \theta < 140^\circ$)



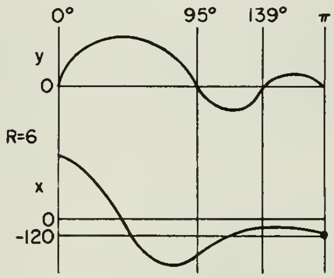


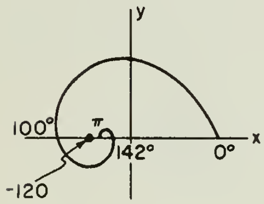
1 root (real) @ R = 6

$\theta = \pi \quad A_\zeta = 0^\circ$

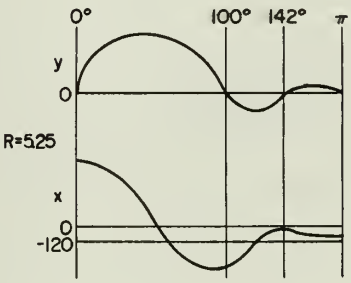
2 roots @ R < 6

If a complex pair exists it should be found in the vicinity of $\theta = 139^\circ$

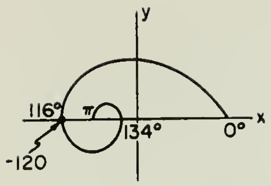


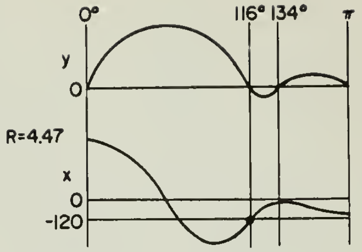


2 roots @ R< 5.25

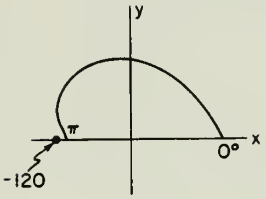


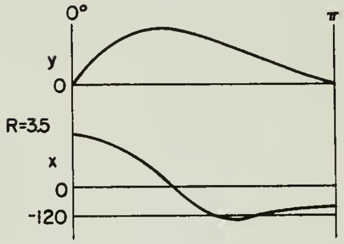
Example 3 (cont.)





2 roots @ R = 4.47
θ = 116° A_ζ = 64°





No roots @
R ≤ 3.5

The polynomial

$$P(z) = z^3 + 10\,z^2 + 44z + 120$$

has roots $z = -\ 6 \qquad z = -\ 2 \pm 4j$

$$(\,w_n = 6^{\cdot},\ A_\zeta = 0^\circ)\ (w_n = 4.47,\ A_\zeta = 60^\circ\,)$$

The factored form is therefore

$$P(s) = (z + 6)(z + 2 + 4j)(z + 2 - 4j)$$

The accuracy of θ indicated by the synthesizer is very good except when θ approaches 180°.

Figure 8 shows all the polynomial contours sketched above. It was constructed by taking the values from the tapes to provide a series of superimposed polynomial contours for all values of R.

Another series of curves was constructed by connecting points of equal θ in order that constant θ curves for varying R may be obtained. Cauch's Residue Theorem is applied to the constant θ curves. The encirclements are counted and the θ values determine the sector in which roots, if any, are located.

Example 3

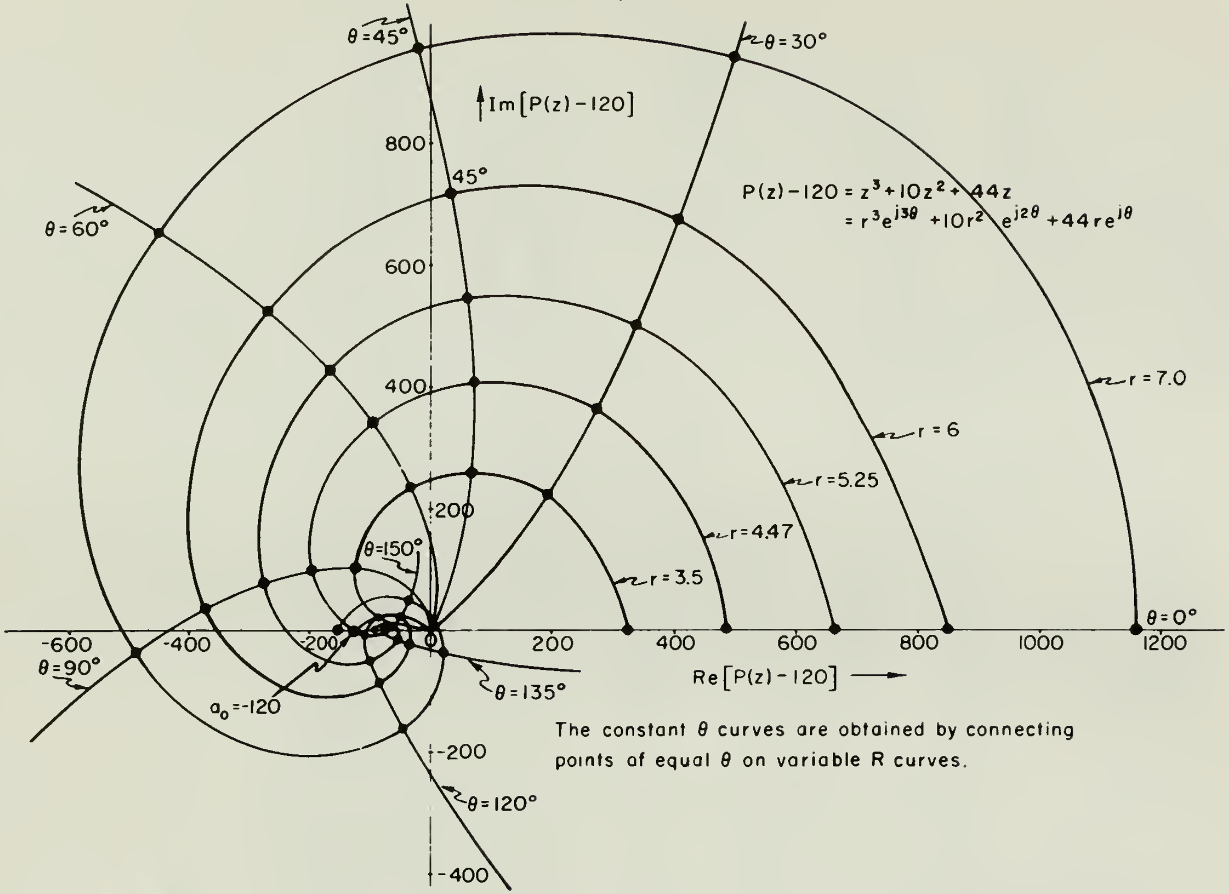


Figure 8: Polynomial Plane Plot of Constant R − Variable θ and Constant θ − Variable R for Equation of Third Degree.

EXAMPLE 3

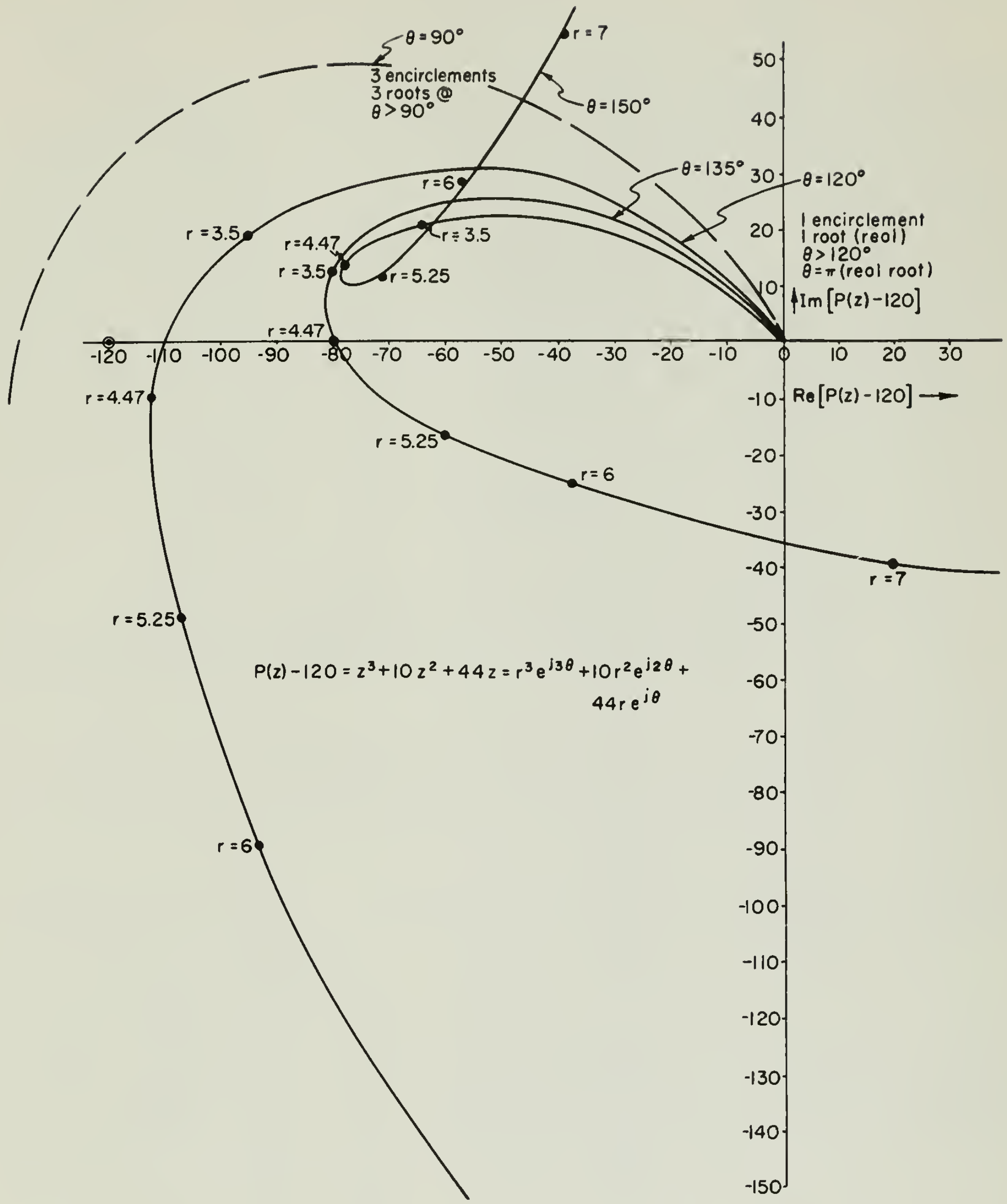


Figure 9: Expanded Portion of Region Near Origin in Figure 8 for Constant θ - Variable R.

TABLE 1

Tabulation of Synthesizer Scale Settings for Example 1

Calculation of Powers of R				Calculation of Coefficients					Synthesizer Scale Settings					Constant Term
R	R^2	R^3	R^4	R^5	$13 R^4$	$60 R^3$	$120 R^2$	124 R	S_5	S_4	S_3	S_2	S_1	$a_o^{\square} \approx 52$
6	36	216	1,296	10,176	16,830	12,960	4,320	744	.605	1.0	.770	.257	.0442	.0325
$5.1 = \sqrt{26}$	26	132.6	676	3,450	8,790	7,950	3,120	632	.393	1.0	.894	.355	.0720	.0621
3.5	12.3	43	151.3	529	1,965	2,580	1,475	434	.205	.762	1.0	.572	.168	.212
$\sqrt{2}$	2	2.83	4	5.66	52	170	240	175.5	.0236	.217	.708	1.0	.731	2.27
1.2	1.44	1.73	2.08	2.50	27.1	104	173	149	.01445	.1567	.602	1.0	.862	3.15
.1	1	1	1	1	13	60	120	124	.00806	.1048	.484	.967	1.0	4.4
.5	.25	.125	.0625	.03125	.812	7.5	30	62	.000505	.0131	.121	.484	1.0	8.81

Sample calculations for Scale Settings:

For $R = 6$,

$13 R^4 = 16,830$ (largest value at $R = 6$)

$S_4 = 1.0$

$60 R^3 = 12,960$ $\qquad \frac{12960}{S_3} = \frac{16,830}{1.0}$, $S_3 = .770$

Sample Calculation for position of Constant Term on x axis:

$a_o = 52$, $R=6$

Full scale (1, 0) on Synthesizer = sweep of 10.5 squares

$\frac{16,830}{10.5^{\square}} = \frac{52}{a_o^{\square}}$ $\qquad a_o = .0325$ squares on tape

EXAMPLE 4: Adjustable Coefficient Case

Example 4 illustrates one analytical method of investigating a typical equation with adjustable coefficients in order that the limits of the component parameters affecting the coefficient may be determined which meet the specified dynamic conditions. This method as explained in Example 4 utilizes the algebraic relations between the adjustable coefficients and the sums and products of the resultant roots. This method is suitable for lower-order equations (second or third) with one or two adjustable coefficients.

EXAMPLE 4

A typical characteristic equation, of a positional servomechanism with modified velocity signal damping, is:

$$'p^2 + \left[2\,\zeta_{(ps)} \frac{2\pi\, r_v\, 'p}{1 + 2\pi\, r_v\, 'p} + 2\,\zeta_{(ps)\,(res)} \right] 'p + 1 = 0 \qquad (1)$$

where $'p = \dfrac{p}{\omega_{n\,(ps)}}$, p = Laplace Transform operator

$\omega_{n\,(ps)}$ = undamped natural frequency of positional servomechansim

$\zeta_{(ps)}$ = damping ratio of positional servomechanism

$\dfrac{1}{r_v\, T_{n\,(ps)}} = CT = \dfrac{\tau_v}{T_{n\,(ps)}}$ = non-dimensional characteristic time of signal modifier

Expanding Equation (1):

$$'p^3 + \left[2\zeta_{(ps)} + 2\zeta_{(ps)\,(res)} + \frac{1}{2\pi r_v} \right] 'p^2 +$$

$$\left[2\zeta_{(ps)\,(res)} \frac{1}{2\pi r_v} + 1 \right] 'p + \frac{1}{2\pi r_v} = 0 \qquad (2)$$

Or,

$$'p^3 + (a + c + b)\,'p^2 + (bc + 1)\,'p + b = 0 \qquad (3)$$

where

$$a = 2\zeta_{(ps)}$$

$$b = \frac{1}{2\pi r_v}$$

$$c = 2\zeta_{(ps)\,(res)}$$

For zero forced dynamic error,

$$FDE = \frac{2\zeta_{(ps)\,(res)}}{\omega_{n\,(ps)}} \dot{A}_{(cm)} = 0 \quad \text{for finite } \omega_{n\,(ps)}$$

where $\dot{A}_{(cm)}$ (angular velocity of controlled member) $\neq 0$

Thus, $\zeta_{(ps)\,(res)} = c = 0$

and Equation (3) becomes

$$'p^3 + (a + b)\,'p^2 + 'p + b = 0 \qquad (4)$$

Substituting $'p = \dfrac{p}{\omega_{n\,(ps)}}$, letting $p = \omega_{n\,(r)}\, e^{j\theta}$

and defining $R = \frac{\omega_{n_{(r)}}}{\omega_{n_{(ps)}}}$ and $z = R e^{j\theta}$,

Equation (4) is : $P(z) = z^3 + (a+b) z^2 + z + b = 0$ (5)

If the roots of the polynomial in z $[P(z)]$ are z_1, z_2, and z_3,

$$P(z) = (z - z_1)(z - z_2)(z - z_3) = 0 \quad (6)$$

$$= z^3 - (z_1 + z_2 + z_3) z^2 + (z_1 z_2 + z_2 z_3 + z_1 z_3) z - z_1 z_2 z_3 = 0 \quad (7)$$

For a third order equation with two complex roots, the roots will have the form:

$$z_1 = x_1 = -\alpha$$

$$z_2 = x_2 + jy_2 = -\zeta_r \omega_{n_{(r)}} + j \omega_{n_{(r)}} \sqrt{1 - \zeta_r^2}$$

$$z_3 = x_3 - jy_3 = -\zeta_r \omega_{n_{(r)}} - j \omega_{n_{(r)}} \sqrt{1 - \zeta_r^2}$$

Using these relations and equating coefficients of like terms in Equations (5) and (7) we have:

$$a + b = \alpha + 2 \zeta_r \omega_{n_{(r)}} \quad (8)$$

$$1 = 2 \zeta_r \omega_{n_{(r)}} \alpha + \omega_{n_{(r)}}^2 \quad (9)$$

$$b = \alpha \omega_{n_{(r)}}^2 \quad (10)$$

For the purpose of illustration, the specifications for the dynamic characteristics were taken to be

1) $.4 \leq \omega_{n_{(r)}} \leq 1$

2) $\zeta_r \geq .5$

3) α and $\zeta_r \omega_{n_{(r)}} \geq .2$

<u>Setting $\zeta_r = .5$:</u>

From Equation (8) $\quad a + b = \alpha + \omega_{n_{(r)}}$

$$\text{or} \quad \alpha = (a+b) - \omega_{n_{(r)}}$$

Using Equation (9) $\quad 1 - \omega^2_{n_{(r)}} = \omega_{n_{(r)}} \left[(a+b) - \omega_{n_{(r)}} \right]$

$$\text{or} \quad \omega_{n_{(r)}} = \frac{1}{a+b}$$

Thus $\quad \alpha = a + b - \dfrac{1}{a+b} = \dfrac{(a+b)^2 - 1}{a+b}$

$$\text{and} \quad b = \alpha \cdot \omega^2_{n_{(r)}} = \frac{(a+b)^2 - 1}{a+b} \quad \frac{1}{(a+b)^2} = \frac{(a+b)^2 - 1}{(a+b)^3}$$

Choose $a + b$, calculate b.

For b_{max} : $\dfrac{db}{d\,'t} = - \dfrac{1}{(a+b)^2} + \dfrac{3}{(a+b)^4} = \dfrac{-(a+b)^2 + 3}{(a+b)^4} = 0$

$$\therefore (a+b)\Big|_{b_{max}} = \sqrt{3}$$

$$\text{and } b_{max} = \frac{(\sqrt{3})^2 - 1}{(\sqrt{3})^3} = \frac{2}{3\sqrt{3}} = .385$$

<u>Setting $\omega_{n_{(r)}} = .4$</u>

When $\zeta_r = .5$,

$$a + b = \alpha + .4$$

$$\alpha = (a + b) - .4$$

$$1 - (.4)^2 = 2\,(.5)\,(.4)\left[(a+b) - .4\right]$$

$$a + b = \frac{.84}{.4} + .4 = 2.5$$

$$\alpha = 2.5 - .4 = 2.1$$

$$b = (2.1)\,(.4)^2 = .336$$

When $\zeta_r = 1$.

$$a + b = \alpha + .8$$

$$\alpha = (a + b) - .8$$

$$1 - .4^2 = 2\,(1)\,(.4)\left[(a + b) - .8\right]$$

$$a + b = \frac{.84}{.8} + .8 = 1.85$$

$$\alpha = 1.85 - .8 = 1.05$$

$$b = (1.05)\,(.4)^2 = .168$$

Other points between these limits are calculated similarly by using other values of $.5 \leq \zeta_r \leq 1$

For $\alpha = .2$

$$b = \alpha\,\omega^2_{n_{(r)}} = .2\,\omega^2_{n_{(r)}}\,, \quad \omega^2_{n_{(r)}} = 5\,b$$

$$1 - \omega^2_{n_{(r)}} = 2\,\zeta_r\,\omega_{n_{(r)}}\,(.2)$$

$$2\,\zeta_r\,\omega_{n_{(r)}} = \frac{1 - \omega^2_{n_{(r)}}}{.2} = 5\,(1 - \omega^2_{n_{(r)}})$$

$$a + b = .2 + 2\,\zeta_r\,\omega_{n_{(r)}} = .2 + 5\,(1 - \omega^2_{n_{(r)}})$$

$$= .2 + 5\,(1 - 5\,b)$$

$$= .2 + 5 - 25\,b$$

$$a + b = 5.2 - 25\,b$$

$$b = \frac{5.2 - (a + b)}{25}$$

Choose values of $a + b$, calculate b.

For $\zeta_r = 1$

$$a + b = \alpha + 2\,\omega_{n_{(r)}}\,, \quad 1 - \omega^2_{n_{(r)}} = 2\,\omega_{n_{(r)}}\left[(a+b) - 2\omega_{n_{(r)}}\right]$$

$$b = \alpha\,\omega^2_{n_{(r)}} = \left[(a+b) - 2\,\omega_{n_{(r)}}\right]\omega^2_{n_{(r)}}$$

$$= \frac{1 - \omega^2_{n_{(r)}}}{2\,\omega_{n_{(r)}}}\,\omega^2_{n_{(r)}}$$

$$b = \left[\frac{1 - \omega_{n(r)}^{2}}{2} \right] \omega_{n(r)}$$

$$a+b = \frac{1 - \omega_{n(r)}^{2}}{2\,\omega_{n(r)}} + 2\,\omega_{n(r)} = \frac{1 + 3\,\omega_{n(r)}^{2}}{2\,\omega_{n(r)}}$$

Assume $0 \leq \omega_{n_r} \leq 1$, calculate corresponding $a+b$ and b on $\zeta_r = 1$ line.

From the preceding calculations, Figure 10 was constructed, which shows the possible combinations and variations of the parameters (a+b) and (b) which satisfy the performance specifications.

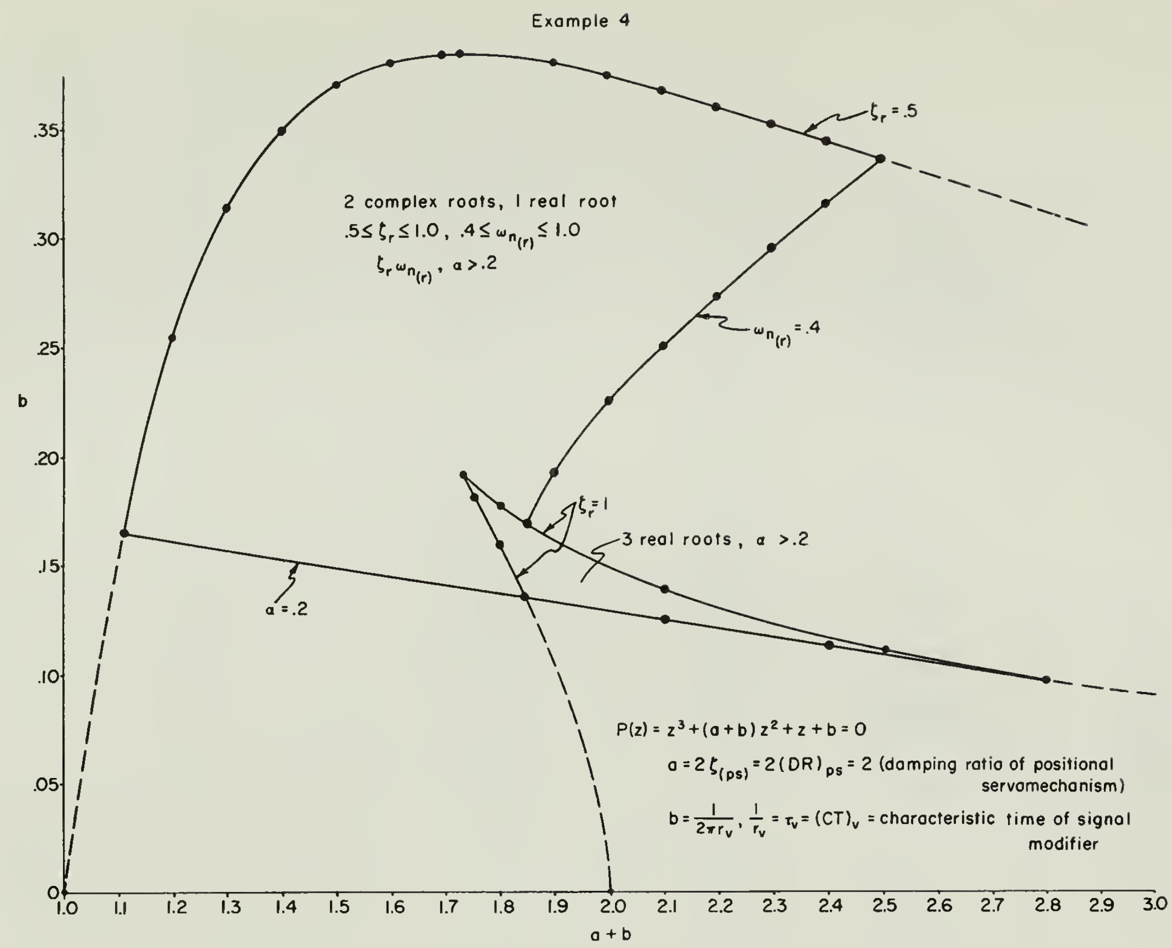


Figure 10: Permissible Variations of Parameters (a+b) and (b) Which Satisfy Performance Specifications.

The complex plane root loci in Figure 11 may be constructed, with the variables (a+b) and (b), as shown below:

$$P(z) = z^3 + (a+b)\, z^2 + z + b = 0$$

Relating equations:

$$a+b = \alpha + 2\zeta_r \omega_{n(r)} \tag{a}$$

$$1 = \omega_{n(r)}^2 + 2\zeta_r \omega_{n(r)} \alpha \tag{b}$$

$$b = \alpha\, \omega_{n(r)}^2 \tag{c}$$

Solving (b) for α :

$$\alpha = \frac{1 - \omega_{n(r)}^2}{2\zeta_r \omega_{n(r)}} \tag{1}$$

Substituting in (a):

$$a+b = \frac{1-\omega_{n(r)}^2}{2\zeta_r \omega_{n(r)}} + 2\zeta_r \omega_{n(r)} \tag{2}$$

Solving for $2\zeta_r \omega_{n(r)}$:

$$2\zeta_r \omega_{n(r)} = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4\,(1-\omega_{n(r)}^2)}}{2} \tag{3}$$

By assuing $0 \le \omega_{n(r)} \le 1.0$, $2\zeta_r \omega_{n(r)}$ can be calculated in terms of a + b.

For each assumed value of $\omega_{n_{(r)}}$, values of a + b may be tabulated, and the corresponding α's and b's found which satisfy the above relating equations.

These tables are presented in Appendix B and the results are plotted in Figure 11.

The $\alpha = .2$ line was obtained by letting $\alpha = .2$ in Equation (a). Solving for $\zeta_r \omega_{n_{(r)}}$:

$$\zeta_r \omega_{n_{(r)}} = \frac{(a+b) - \alpha}{2} = \frac{(a+b) - .2}{2}$$

The points on the $\alpha = .2$ line will lie at the intersection of the a + b = constant curves and the corresponding vertical $\zeta_r \omega_{n_{(r)}}$ lines. This curve is also plotted in Figure 11.

The area enclosed by the $\alpha = .2$ line, the $\zeta_r = .5$ line, the $\zeta_r = 1.0$ line, and $4 \leq \omega_{n_{(r)}}$ is the area in which the values of a + b and b will satisfy the specifications of the problem.

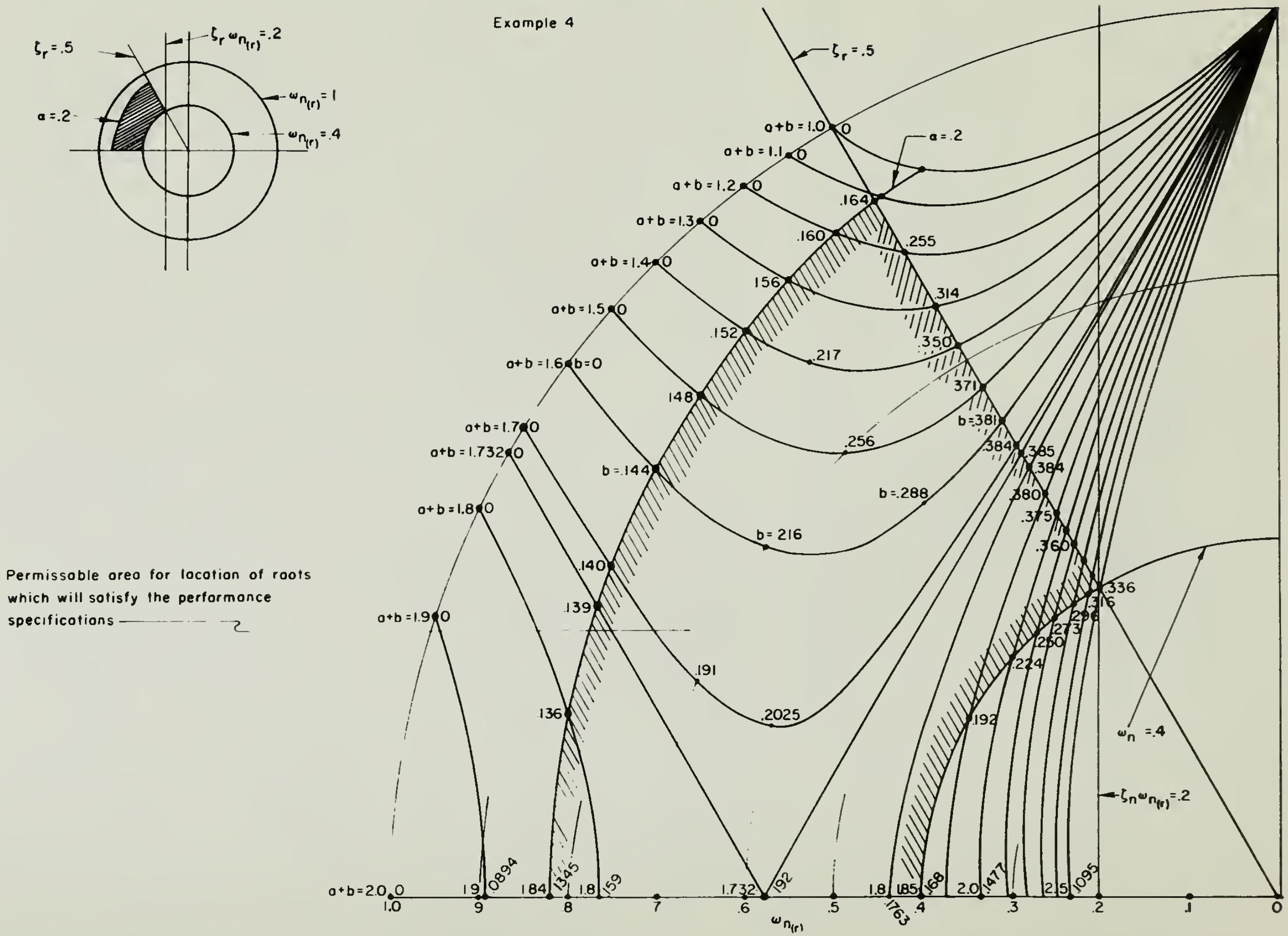


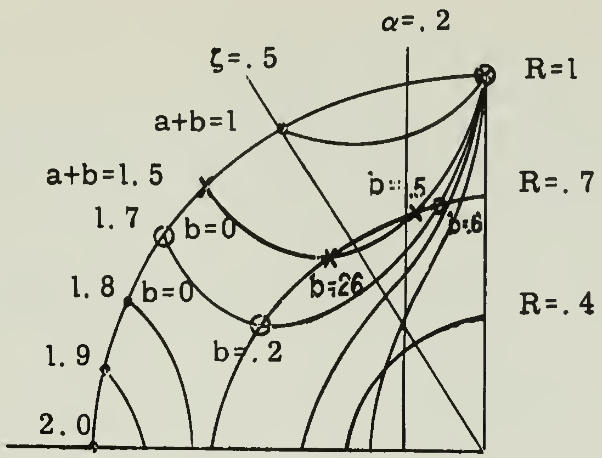
Figure 11: Constant a + b – Variable b Loci Plotted on Root Complex Plane

One might also construct such a plot as Figure 11 solely through the use of the synthesizer. Runs are made at various radii, (R), and for various values of the second coefficient, (a + b), and reading the values of the constant term, (b), which would give roots at each combination of (R) and (a+b).

This is not recommended, however, since a very large number of runs would be required, with the number of runs required being further increased if there were more adjustable coefficients. Analytical methods such as this, or use of the cubic charts in Volume II,[2] Instrument Engineering would enable one to construct a coefficient boundary plot in less time.

In engineering applications and control system design problems such stringent boundaries may not be required. If the primary requirement is selection of components with parameters (represented by the coefficients) which will permit the system performance to be well within the specifications, use of the synthesizer may greatly expedite selection of coefficients which appear to be satisfactory. Checking this selection on the synthesizer will insure that the performance is within specifications.

In this instance only two runs on the synthesizer, one at a radius, (R), of about 1.0 and one at a radius of about 0.7 for an (a + b) of around 1.5 will be sufficient to give a rough but adequate locus of complex roots on the root plane. Two more runs at an (a + b) of about 1.7 at the same radii will give another locus. These four runs and two resultant loci will be sufficient to give a general idea of the loci of other selected values of the coefficient. Figure 12 shows such an estimated plot which corresponds to the computed plot of Figure 11.



Approximate Root Plane Plot of Example 4

Figure 12

From this rough root plane locus it is apparent that a value of (a+b) of 1.6 with a constant term (b), of 0.2 should be one of many possible combinations which give a satisfactory solution well within specifications. There remains only to set up this equation to check the proposed selections. Few trial and error runs will be required to locate the roots accurately, since a glance at the rough root plane plot will show that the complex pair should be found at an ω_n, of about 0.65 or 0.7. A rudimentary knowledge of Root Locus will indicate that the real root should be found at a radius of about 0.5 since for this equation the real root moves left from the origin at twice the rate the x component of the conjugate pair moves right. The Root Locus method is described in Volume II, Instrument Engineering [2].

As may be seen from Figure 10 giving the boundary values of (a+b) and (b) which yield acceptable solutions, the three adjustable parameters, a, b, and c, reduce to two adjustable coefficients. It is shown that boundary values of two adjustable coefficients may be presented on a two-axis or two-dimensional plot. It follows that with three adjustable coefficients, which may be made up of any number of adjustable system parameters, a cubic or three-dimensional plot would be required.

It is apparent that with more than two or three adjustable coefficients, the presentation of definitive boundary conditions for each coefficient when dependent upon each of the others becomes impractical. It was also found that a method based on the algebraic relation between adjustable coefficients and resultant roots becomes unwieldy for fourth order or higher equations.

i.e., 1^{st} coefficient $= 1$

$$2^{nd} \text{ coefficient} = \sum_{i=1}^{n} z_i$$

$$n\text{th coefficient} = \prod_{i=0}^{n} z_i$$

Accordingly, it appears that the most generally suitable method of investigating the result of adjustable parameters, and hence adjustable coefficients, will be to set up a series of inequalities for various conditions and observing the effect upon the resultant performance. From these inequalities reasonable engineering estimates may be made for the selection of various sets of coefficients, and the resultant dynamic characteristics that each set should yield. These estimates are readily checked on the machine in the manner given for the fixed coefficient case to insure that they do in fact give performances close to that which was anticipated. These checks performed on the machine give, in addition to rather precise dynamic characteristics for the coefficients selected, a good indication of the effect of certain other variations in selected coefficients. This technique will be illustrated in Example 5.

CHAPTER 5

MODIFICATIONS INSTALLED ON THE FOURIER SYNTHESIZER

The considerable labor required in cross plotting from the x and y tapes to the P (z) plane may be eliminated by the use of a complex - vector adder connected to x and y signal leads. A servo driven two-axis recorder was used for this purpose. Timed pulses were introduced on the x and/or y axis to provide an indication of selected values of θ. The harmonic synthesizer sweeps out from 0° to 360° at a uniform rate and θ is therefore proportional to time.

A more satisfactory method of observing and recording the polynomial contour (P (z) plane) plot was found by utilizing a standard cathode ray oscilloscope in place of the two-axis recorder, and a Polaroid Camera for recording. The associated circuitry and additional components required to modify the harmonic synthesizer to provide a suitable computer are described in Appendix A. This resulted in a much smoother and more accurate plot since the mechanical and servo response difficulties inherent in the two axis recorder were eliminated. Also, since the area of primary interest on the P (z) plane is in the vicinity of the x axis cross-over point (where y = 0), this area can be enlarged to cover the entire scope. Great latitude is possible in scaling both the synthesizer and scope, permitting substantial flexibility in selecting the regions to be more closely examined. Through the use of the Polariod camera and multi-exposure, as many runs as necessary can be superimposed on one record for comparison. The sweep speed of the trace is slow enough so that it is practicable to follow the spot with a grease pencil directly on the face of the scope or on tracing paper. Thus the camera may not be required unless a permanent accurate record is required.

This superposition of record technique is the only practical method by which curves for constant θ and varying R can be obtained, since the synthesizer is inherently a constant R and variable θ device.

The set of constant θ curves are obtained merely by connecting points of equal θ on the several constant R curves. For these records the synthesizer is not scaled for maximum amplitude for each run of different radii, but scaled as shown in Example 5 to provide the actual relationship of each radius plot to all the others.

θ indication was provided by fabrication and installation of a cam on the synthesizer drive shaft as described in Appendix A. A microswitch actuated by the cam follower was installed on the synthesizer and the signal produced was used to blank out the scope for about 4°, thus providing a θ indication. Cam teeth were cut at $\theta = 90^{\circ}$, 120°, 135°, and 150°, and the cam was cut to blank out the scope from 180° to 360°, since the trace is symmetrical about the x axis and the last 180° would contribute nothing further.

The x and y axes on the scope trace may be obtained simply by turning off the y and x scope amplifiers respectively. The x and y axes, being linear scales, may be calibrated in a manner identical to the method of calibrating the component tapes or as shown in Example 5. Thus, the x axis location of the constant term, (encirclement point) may be determined. The dynamic characteristics or roots of an equation with constant coefficients may be found to a satisfactory degree of accuracy quickly and easily without using large amounts of Sanborn Recorder tape. Various radii (R) runs are scaled and set into the synthesizer. Observing each trace for encirclements and the x axis cross-overs relative to the constant term on the x axis serves as a good estimate for the radius of the next run. Scaling and setting values into the machine requires very little time or effort. All roots of the equation may be found in this manner. A root is indicated when the trace passes through (or near, depending on the degree of accuracy desired) the encirclement point. The radius (R) used gives the ω_n of the root, and the θ marks indicate the polar angle of the root, measured counter-clockwise from π on the root plane $\left[(ADR) = \pi - \theta\right]$, as well as the damping ratio $\left[\zeta = \cos (ADR)\right]$. No Sanborn tape need be used unless desired. When a more precise value of (ADR) is required than that obtainable from the five θ indications provided by the cam, a single tape can be run for the root value of the radius R and a more accurate value of θ obtained.

For higher order equations, fifth or higher, it may be desirable to enlarge the scope presentation, particularly when searching for roots at the smaller radii, so that the immediate vicinity of the encirclement point occupies the entire presentation. When the origin no longer appears at the far right side of the scope, as with an enlarged presentation, it may be necessary to re-run previously scaled plots where the x cross overs are known in order to re-scale the portion of the negative real axis that appears on the presentation. With some higher order equations it may be advantageous to divide through by the constant term prior to making any runs, thus shifting the origin to the constant term on the negative real axis. Thus the origin becomes the encirclement point and calibrating the x axis is no longer a problem.

The principles, procedures and equipment just discussed are particularly well adaptable to equations with fixed coefficients and an adjustable constant term. Problems of this type arise frequently, particularly in control system and servo problems involving an adjustable open loop sensitivity.[6] Runs are made as previously discussed for a few radii and the ω_n and (ADR) at x cross-overs are plotted on the root plane. The value of the x axis intercept is the constant term for the equations thus formed with roots at the ω_n and ζ indicated by the cross-over. Ordinarily just a few such runs, with their corresponding roots and value of the constant term will be sufficient to fair in a reasonably accurate root locus on the root plane plot. The effect on the magnitude and (ADR) of the roots of varying the constant term (or open loop gain) may be determined from the tape or photo record.

The analysis of the case where all the coefficients and the constant term are adjustable will be illustrated by an example in the next chapter. This case arises when system parameter specifications are to be set which will enable the resultant system to meet the required dynamic characteristics.

CHAPTER 6

ILLUSTRATION OF THE ANALYSIS TECHNIQUE

EXAMPLE 5: Adjustable Coefficient Case

A typical performance equation of a lead modified positional servomechanism is

$$\left[I p^2 + c_d \; p + \frac{1+\gamma\tau p}{1+\tau p} S_{(ps)} \right] A_{cm} = \frac{1+\gamma \; \tau \; p}{1+\tau \; p} S_{(ps)} A_{in} + M_{(intf)} \quad (1)$$

where I = equivalent moment of inertia, referred to the controlled member

c_d = damping coefficient, referred to the controlled member

$\frac{1+\gamma \tau p}{1+\tau p}$ = performance function of the lead modifier network, where $\gamma < 1$.

τ = characteristic time of the lead modifier

$S_{(ps)}$ = sensitivity of the positional servomechanism for angle in, torque out.

M_{intf} = interference torque

A_{cm} = angular position of the controlled member

A_{in} = command input angle

$p = \frac{d}{dt}$ = Laplace Transform operator

If we define: $\frac{I}{\gamma\, S_{(ps)}} = \frac{1}{\omega^2_{n(ps)}}$, $\frac{c_d}{\gamma\, S_{(ps)}} = \frac{2\,\zeta_{(ps)}}{\omega_{n(ps)}}$

$'p = \frac{p}{\omega_{n(ps)}}$ and $r = \frac{\tau}{T_{n\,(ps)}}$

Equation (1) becomes

$$\left\{ 'p^2 + 2\,\zeta_{(ps)}\ 'p + \frac{1+\gamma\, 2\pi r\, 'p}{\gamma\,(1+2\,\pi\, r\, 'p)} \right\} A_{out} = \frac{1+\gamma\, 2\,\pi\, r\, 'p}{\gamma(1+2\,\pi\, r\, 'p)} A_{in} \qquad (2)$$

$$\text{so} \left\{ 'p^3 + \left[2\,\zeta_{(ps)} + \frac{1}{2\pi r} \right] 'p^2 + \left[\frac{2\,\zeta_{(ps)}}{2\pi r} + 1 \right] 'p + \frac{1}{\gamma}\,\frac{1}{2\pi r} \right\} A_{out}$$

$$= \left\{ \frac{1}{\gamma} \left[\frac{1}{2\pi r} + \gamma\, 'p \right] \right\} A_{in} \qquad (3)$$

The characteristic equation is:

$$'p^3 + \left(2\,\zeta_{(ps)} + \frac{1}{2\pi r}\right) 'p^2 + \left(\frac{2\zeta_{(ps)}}{2\pi r} + 1 \right) 'p + \frac{1}{\gamma}\,\frac{1}{2\pi r} = 0 \qquad (4)$$

let $'p = z = R\, e^{j\theta}$, $2\,\zeta_{(ps)} = a$, $\frac{1}{2\pi r} = b$, $\gamma = c$

Then $$P(z) - \frac{b}{c} = z^3 + (a+b)\, z^2 + (ab+1)\, z \qquad (5)$$

or $$P(R,\theta) - \frac{b}{c} = R^3\, e^{j3\theta} + (a+b)\, R^2\, e^{j2\theta} + (ab+1)\, R\, e^{j\theta} \qquad (6)$$

Taking reasonably broad limitations based on servo principles as a starting point for the analysis:

$$.5 \lessgtr \zeta_{(ps)} \lessgtr 1.5 \qquad .05 \lessgtr r \lessgtr .5 \qquad .1 \lessgtr \gamma \lessgtr 1$$

Hence: $1 \lessgtr a \lessgtr 3$ $\quad .3 \lessgtr b \lessgtr 3$ $\quad .1 \lessgtr c \lessgtr 1$

and $\quad .3 \lessgtr ab \lessgtr 9$

Therefore: $1.3 \lessgtr a + b \lessgtr 6$ $\quad 1.3 \lessgtr ab + 1 \lessgtr 10$ $\quad .1 \lessgtr c \lessgtr 1$

Values of the second coefficient, $a+b$, were picked which covered the range of variation of $a+b$; i.e. $a+b = 1.3,\ 3.5$, and 6.

However, choosing a value for $a+b$ restricts the possible variation of the third coefficient, $ab+1$, since b must remain a real number physically.

Thus, if $a+b = 1.3$,

$$a = 1.3 - b$$

and

$$ab+1 = x = (1.3-b)\,b+1$$

$$= -b^2 + 1.3\,b + 1$$

Transposing and solving for b,

$$b = \frac{1.3 \pm \sqrt{1.69 - 4(x-1)}}{2} \tag{7}$$

The maximum value for x for which b remains a real number is obtained by setting the quantity under the radical equal to zero and solving for x:

$$4\,(x-1) = 1.69$$

$$x = \frac{1.69}{4} + 1 \doteq 1.42$$

Therefore, $1.3 \le ab+1 \le 1.42$ when $a+b=1.3$

The value of b is determined from Equation (7) by the substitution $x = 1.42$.

Similarly, the permissible variations of ab+1 when a+b=3.5 and a+b=6 were determined and found to be

$1.3 \le ab+1 \le 4.06$ when $a+b = 3.5$

and $1.3 \le ab+1 \le 10$ when $a+b = 6$

On the basis of the preceding estimates, runs were made for various R's with the following adjustable coefficient settings:

Run	a+b	ab+1	$\frac{b}{c}$
1-1	1.3	1.3	$.3 \le \frac{b}{c} \le 3$
1-2'	1.3	1.42	$.65 \le \frac{b}{c} \le 6.5$
2-1	3.5	1.3	$.3 \le \frac{b}{c} \le .9$
2-2'	3.5	4.06	$1.75 \le \frac{b}{c} \le 17.5$
3-1	6	1.3	$.3 \le \frac{b}{c} \le .5$
3-2	6	10	$3 \le \frac{b}{c} \le 30$

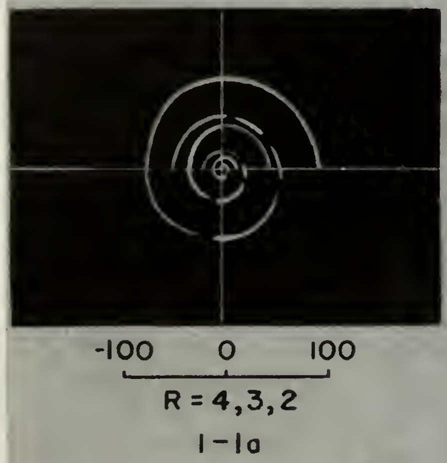
These runs were photographed and are presented on the succeeding pages. The conclusions drawn from the examination of the photographs appear with the records.

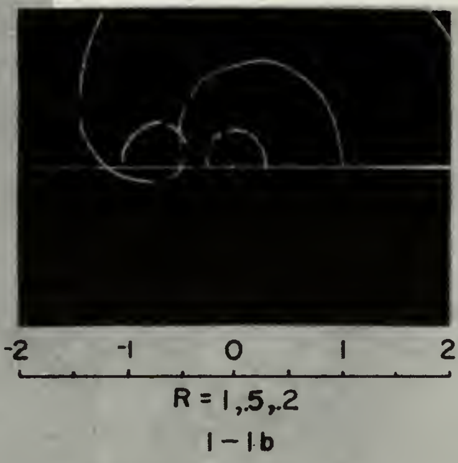
Figures 13, 14, 15, 16, 17 and 18 For Example 5 Follow.

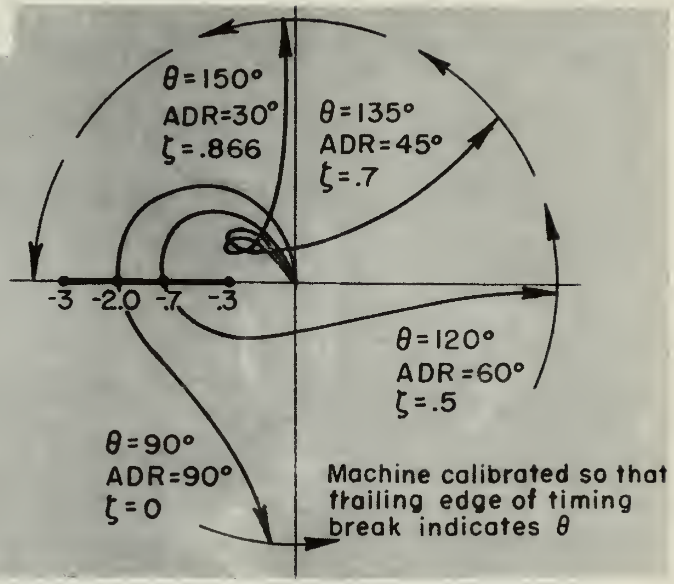
$$P(z) = z^3 + (a+b)\,z^2 + (ab+1)\,z + \frac{b}{c} = 0$$

$a+b = 1.3 \qquad ab+1 = 1.3 \qquad b = .3$

$.3 \leq \frac{b}{c} \leq 3$







$3 \geq \frac{b}{c} \geq 1$	1 complex pair	$1.8 \geq \omega_n \geq 1$
	1 real root	$1.5 \geq \alpha \geq 1$
$1 \geq \frac{b}{c} \geq .5$	1 complex pair	$1 \geq \omega_n \geq .5$
	1 real root	$1 \geq \alpha \geq .5$
$.5 > \frac{b}{c} \geq .3$	1 complex pair	$2 > \omega_n \geq 1$
	1 real root	$.5 \geq \alpha \geq .2$

$\frac{b}{c} > 2$	No complex pairs	$ADR < 90^\circ$ $(\zeta > 0)$
	Unstable, 1 complex pair in right half plane	
$2 \geq \frac{b}{c} \geq .7$	1 complex pair	$60^\circ \leq ADR \leq 90^\circ$ $(.5 \geq \zeta \geq 0)$
	1 real root	$ADR = 0$ $(\zeta = 1)$
$.7 \geq \frac{b}{c} \geq .3$	1 complex pair	$45^\circ < ADR \leq 60^\circ$ $(.7 > \zeta \geq .5)$
	1 real root	$ADR = 0$ $(\zeta = 1)$

Most nearly satisfactory conditions with a max. ζ_n would be attained with a $\frac{b}{c} \doteq .5$

$\omega_n \doteq 1 \qquad \zeta_n \doteq .6$

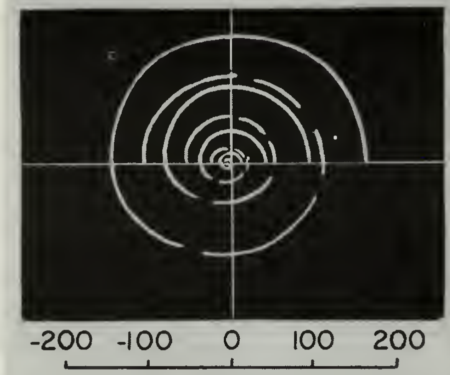
$\alpha \doteq .5$

Fig. 13 Example 5

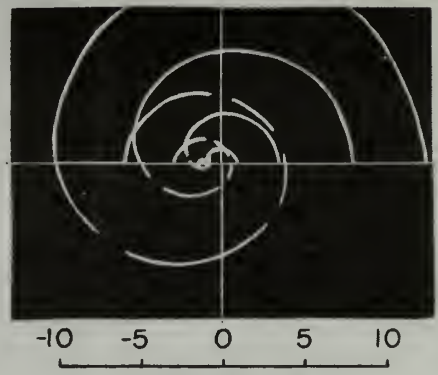
$$P(z) = z^3 + (a+b)\,z^2 + (ab+1)\,z + \frac{b}{c} = 0$$

$a+b = 1.3 \qquad ab+1 = 1.42 \qquad b = .65$

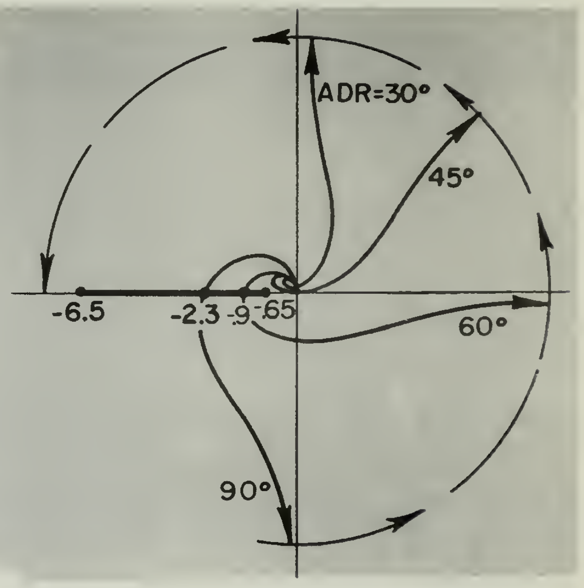
$$.65 \le \frac{b}{c} \le 6.5$$



R=5,4,3,2
1-2'a



R=2,1.5,1,.5
1-2'b



Because of ζ condition, no $\frac{b}{c} > 2.3$ will be considered ($\frac{b}{c} > 2.3$, unstable)

$2.3 \geq \frac{b}{c} \geq .8$	1 complex pair	$1 \geq \omega_n \geq .5$
	1 real root	$1 \geq \alpha \geq .5$
$.8 > \frac{b}{c} \geq .65$	1 complex pair	$1.3 > \omega_n \geq 1$
	1 real root	$.5 \geq \alpha$

$\frac{b}{c} > 2.3$	No complex pairs (Unstable)	ADR $< 90^\circ$ ($\zeta > 0$)
$2.3 \geq \frac{b}{c} \geq .9$	1 complex pair	$60^\circ \le$ ADR $\geq 90^\circ$ ($.5 \geq \zeta \geq 0$)
	1 real root	($\zeta = 1$)
$9 \geq \frac{b}{c} \geq .65$	1 complex pair	$45^\circ \le$ ADR $\le 60^\circ$ $.7 \geq \zeta \geq .5$
	1 real root	($\zeta = 1$)

Most nearly satisfactory conditions occur with a $\frac{b}{c} \doteq .85$

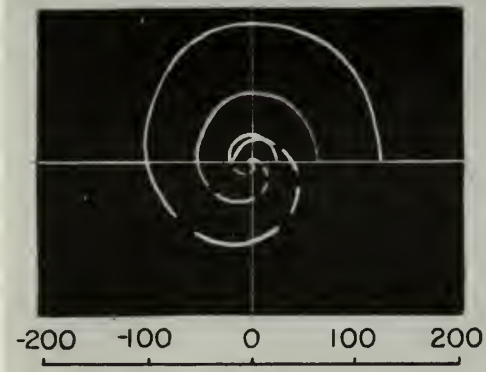
$\omega_n \doteq .8 \qquad \zeta \doteq .55$

Fig. 14 Example 5

$$P(z) = z^3 + (a+b)\,z^2 + (ab+1)\,z + \frac{b}{c} = 0$$

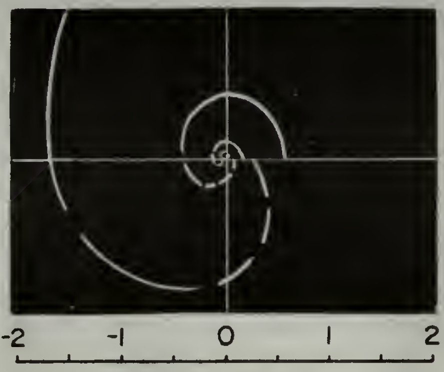
$a+b = 3.5 \qquad ab + 1 = 1.3 \qquad b = .09$

$$.09 \le \frac{b}{c} \le .9$$



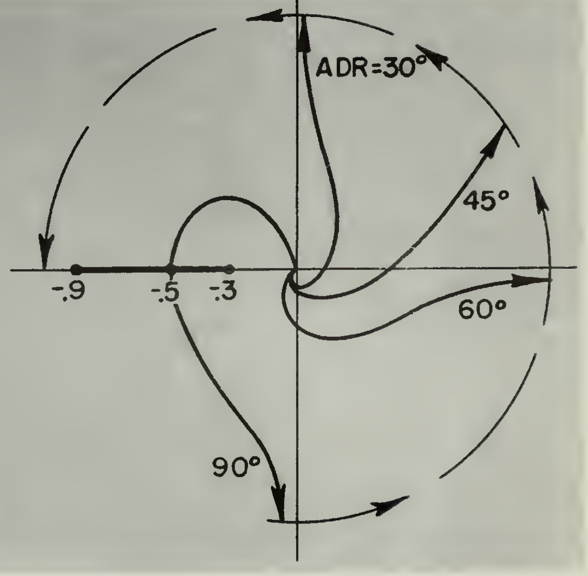
R=4,3,2

2-1a



R=2,1,.5,.2

2-1b



$.5 \geq \frac{b}{c} \geq .3$ 1 complex pair $1 \geq \omega_n \geq .5$

1 real root $4 > \alpha > 2$

This set of coefficients is not likely to be of any value since the maximum ζ for the complex roots is only about .2

$\frac{b}{c} > .5$ No complex pair (Unstable) ADR < 90° ($\zeta > 0$)

$.5 \geq \frac{b}{c} \geq .3$ 1 complex pair 75° < ADR < 90°

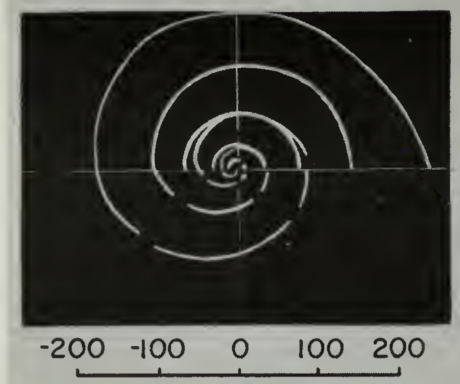
The region $\frac{b}{c} < .3$ could be expanded on the scope and examined in some detail but since the practical $\frac{b}{c}$ min. was taken as .3, this is not necessary.

Fig. 15 Example 5

$$P(z) = z^3 + (a+b)\,z^2 + (ab+1)\,z + \frac{b}{c} = 0$$

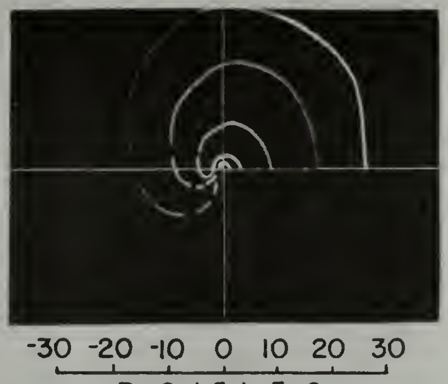
$a+b = 3.5 \qquad ab + 1 = 4.06 \qquad b = 1.75$

$$1.75 \le \frac{b}{c} \le 17.5$$



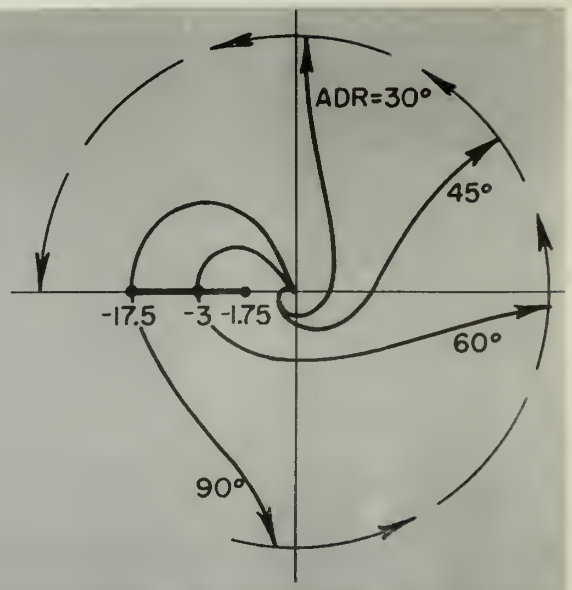
R = 5,4,3,2

2-2'a



R = 2,1.5,1,.5,.2

2-2'b



$17.5 \geqq \frac{b}{c} \geqq 9$	1 complex pair	$2 \geqq \omega_n \geqq 1.5$
	1 real root	$4 > \alpha > 2$
$9 \geqq \frac{b}{c} \geqq 6$	1 complex pair	$1.5 \geqq \omega_n \geqq 1$
	1 real root	$3 > \alpha > 2$
$6 \geqq \frac{b}{c} > 3$	1 complex pair	$1 \geqq \omega_n \geqq .5$
	1 real root	$3 > \alpha > .5$
$3 > \frac{b}{c} \geqq 1.75$	1 complex pair	$3 > \omega_n > 2$
	1 real root	$.5 \geqq \alpha \geqq .2$

$12.5 \geqq \frac{b}{c} \geqq 3$	1 complex pair	$90^\circ > ADR \geqq 60^\circ$
		$0 < \zeta \le .5$
	1 real root	$(\zeta = 1)$
$3 \geqq \frac{b}{c} \le 1.75$	1 complex pair	$60^\circ \geqq ADR > 50^\circ$
		$.5 \le \zeta \le .64$
	1 real root	$(\zeta = 1)$

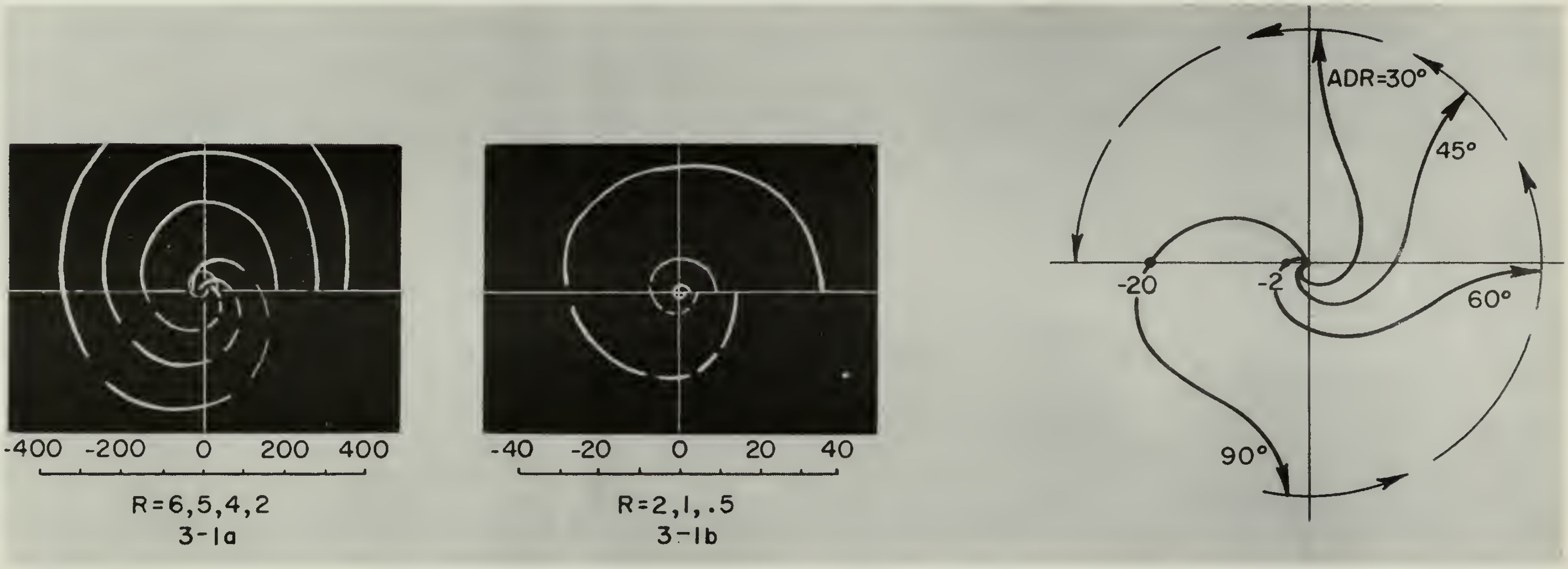
Most nearly satisfactory conditions would be attained with a constant term $\frac{b}{c}$ between 2 and 3.

Fig. 16 Example 5

$$P(z) = z^3 + (a+b)\,z^2 + (ab+1)\,z + \frac{b}{c} = 0$$

$$a+b = 6 \quad ab+1 = 1.3 \quad b = .05$$

$$.05 \leq \frac{b}{c} \leq .5$$



$.5 \geq \frac{b}{c} \geq .3$ 1 complex pair $.5 \geq \omega_n > 0$

1 real root $5 > \alpha > 2$

This can be checked and located more accurately by expanding scope presentation if a more satisfactory set of coefficients cannot be found.

$.5 \geq \frac{b}{c} \geq .3$ 1 complex pair $60^\circ > \text{ADR} \geq 45^\circ$

This region may be examined in greater detail by expanding the scope presentation. However it can be expected that a more satisfactory set of coefficients can be found such that a constant term variation of more than .2 will be possible.

Fig. 17 Example 5

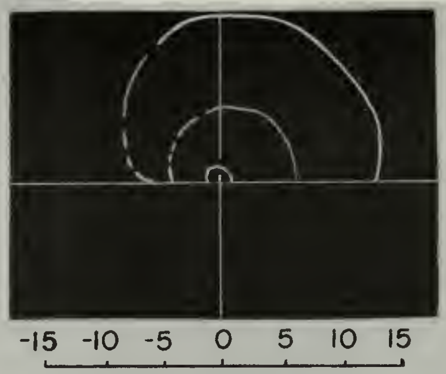
$$P(z) = z^3 + (a+b)\,z^2 + (ab+1)\,z + \frac{b}{c} = 0$$

$a+b = 6 \quad ab+1 = 10 \quad b = 3$

$$3 \leq \frac{b}{c} \leq 30$$

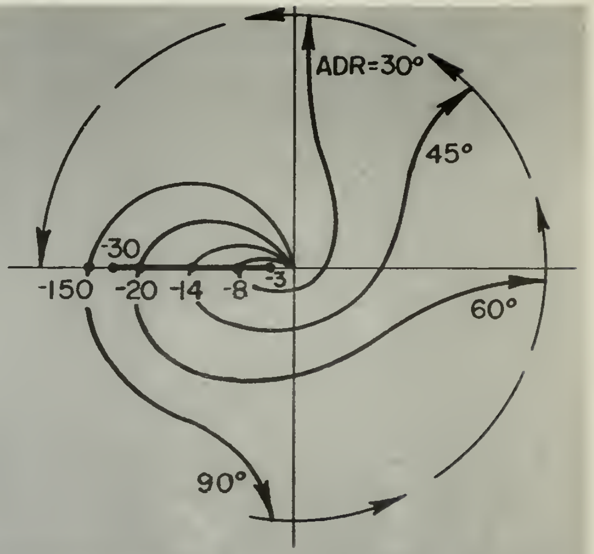
R=6,5,4,2,1

3-2a



R=1,.5,.1

3-2b



$30 \geq \frac{b}{c} \geq 12$	1 complex pair	$2 \geq \omega_n > 1$
	1 real root	$5 \geq \alpha \geq 4$
$12 \geq \frac{b}{c} \geq 9$	1 complex pair	$4 > \omega_n \geq 2$
	1 real root	$2 \geq \alpha \geq .1$
$9 \geq \frac{b}{c} \geq 3$	1 complex pair	$4 > \omega_n > 2$
	1 real root	$1 \geq \alpha \geq .1$

$30 \geq \frac{b}{c} \geq 20$	1 complex pair	$70^\circ > ADR \geq 60^\circ$
$20 \geq \frac{b}{c} \geq 14$	1 complex pair	$60^\circ \geq ADR \geq 45^\circ$
$14 \geq \frac{b}{c} \geq 8$	1 complex pair	$45^\circ \geq ADR \geq 30^\circ$
$8 \geq \frac{b}{c} \geq 3$	1 complex pair	$30^\circ \geq ADR > 0^\circ$
plus	1 real root	$ADR = 0^\circ$

for any permissible value of $\frac{b}{c}$

Selection of coefficients equal to or slighly less than those for this set of runs provide a fairly wide range of dynamic characteristics which should prove satisfactory. The separate parameters (a, b and c) of the system are thus determined by the final selection of the coefficients. Relative transient response time may be controlled by varying $\frac{b}{c}$.

The Damping Ratio of the roots is readily adjustable from $\zeta = .35$ to $\zeta = .86$ by varying $\frac{b}{c}$ from 30 to 9. Still higher damping ratios can be obtained with a small increase in response time by reducing $\frac{b}{c}$ from 9 towards 3. Flexibility of this sort is often of great advantage in practical physical systems.

Fig. 18 Example 5

For any general case where it is desired to select coefficients specifying component parameters, the effect on the dynamic characteristics of varying any coefficient may be readily observed. This is shown in the following series of photographs, Figs. 19 and 20.

The equation used is of the form:

$$P(z) = z^3 + Az^2 + Bz + C = 0$$

A was taken as 1.3, 3.5 and 6.
B was taken as 1.3, 10 and 19.
C is unrestricted and to be selected from whatever specifications are imposed.

It will be observed that this is simply an extension of the estimated boundaries of coefficients of the previous specific problem. Some of the same runs were used to demonstrate that there is nothing unique about any one set of runs. They might be equally applicable to any one of a variety of problems.

For the purposes of illustration, one set of runs will be analyzed for possible conclusions relating the dynamic characteristics and the coefficients. All other sets for any problem are handled in a similar manner.

Photograph 3-3 representing

$$P(z) = z^3 + 6z^2 + 19z + C = 0$$

is sketched in Fig. 21 for clarity although constant θ curves may be drawn directly on the photograph if desired.

Figures 19 and 20 which follow show the effect of the variation of the equation coefficients.

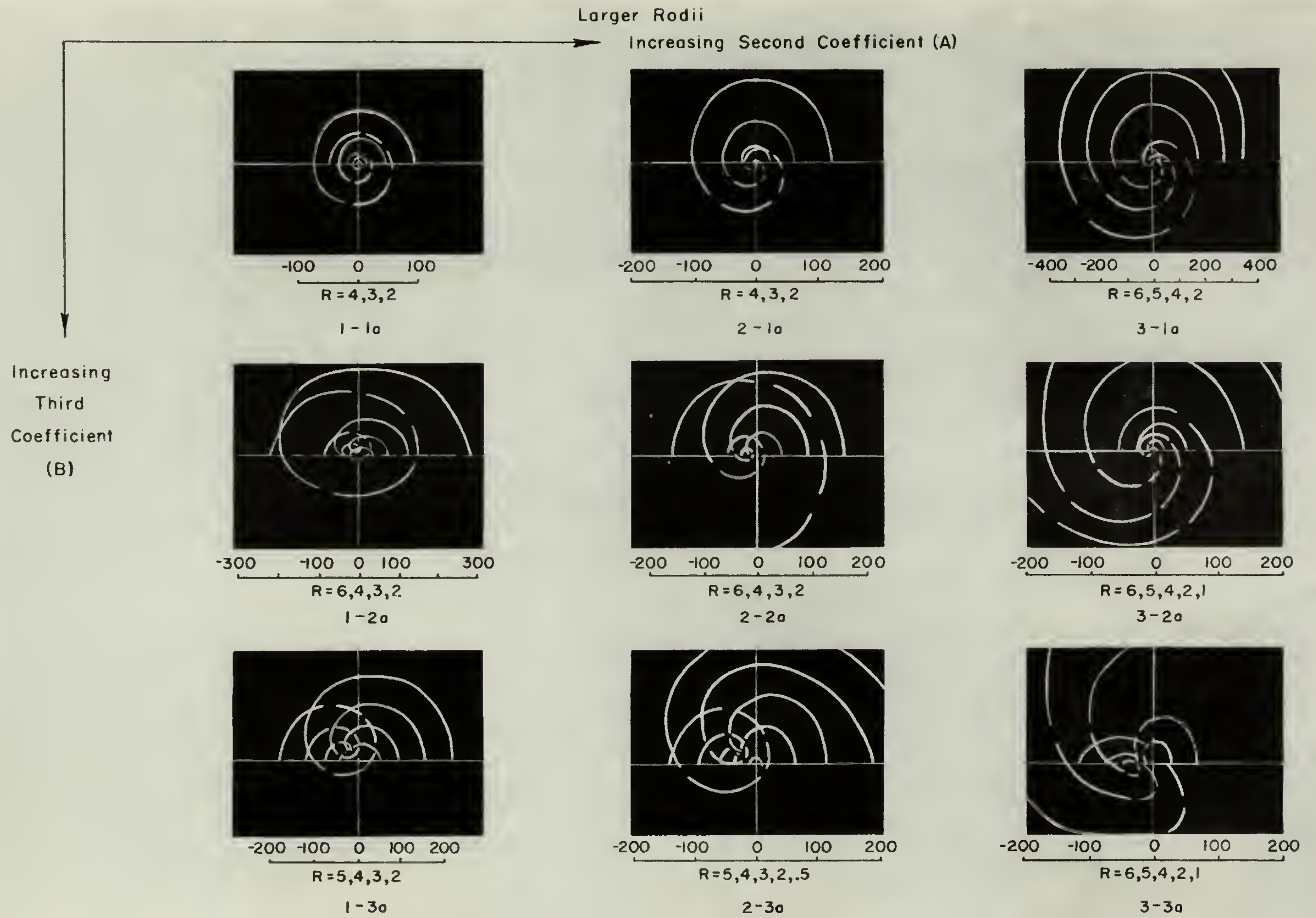


Figure 19: Effect of Coefficient Variation

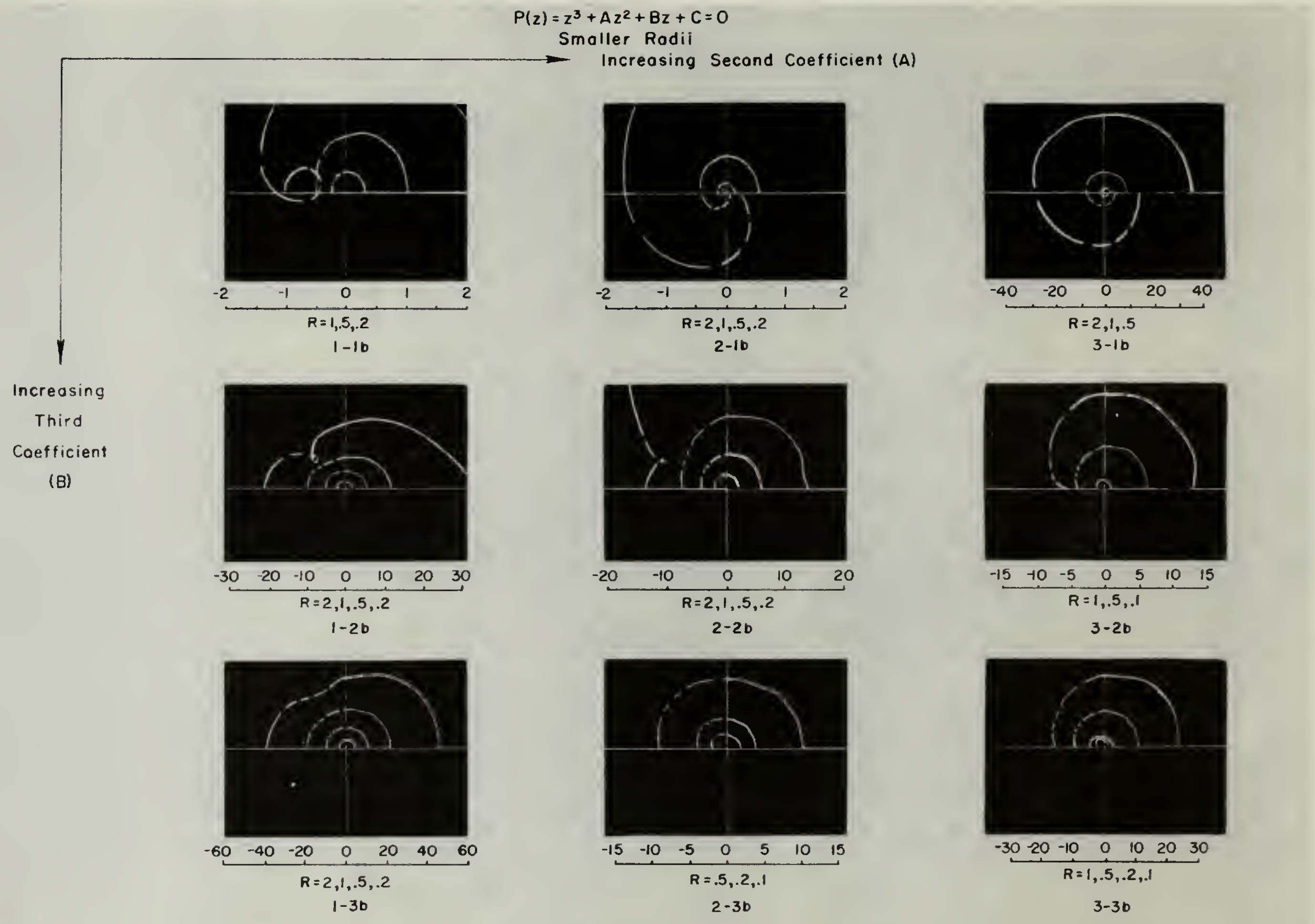
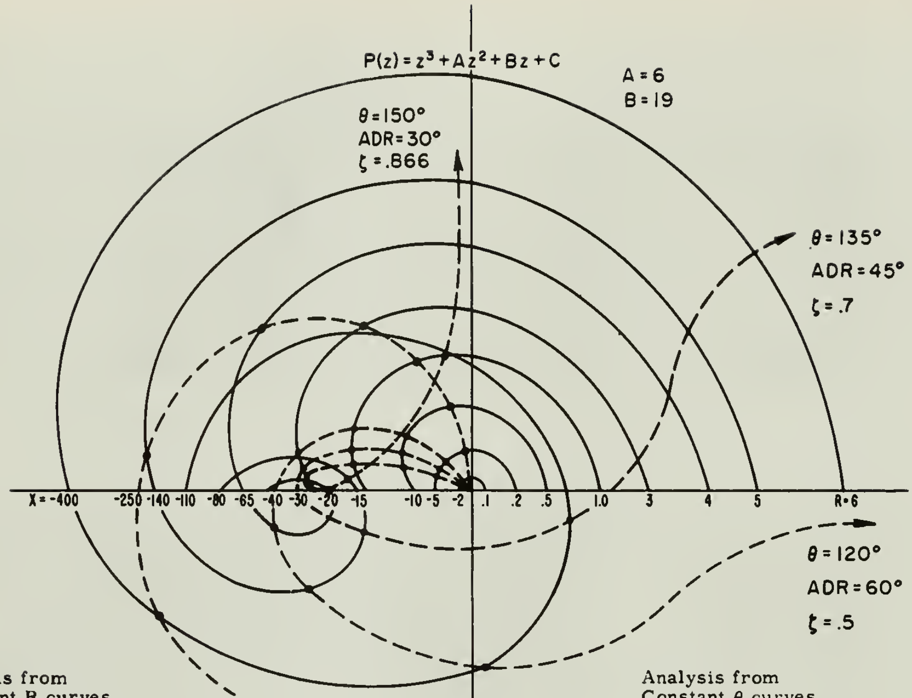


Figure 20: Effect of Coefficient Variation



Analysis from
Constant R curves

$10 \lesssim C \lesssim 15$	1 complex pair	$5 < w_n < 6$
	1 real root	$.5 \lesssim \alpha \lesssim 1.0$
$15 \lesssim C \lesssim 20$	1 complex pair	$4 \lesssim w_n \lesssim 5$
	1 real root	$1 < \alpha < 3$
$25 \lesssim C \lesssim 65$	1 complex pair	$3 < w_n \lesssim 4$
	1 real root	$3 \lesssim \alpha < 5$
$65 \lesssim C \lesssim 140$	1 complex pair	$4 \lesssim w_n \lesssim 5$
	1 real root	$6.5 < \alpha < 4$

Analysis from
Constant θ curves

$0 < C < 20$	1 complex pair	$30^\circ < ADR < 45^\circ$
	1 real root	$ADR = 0$
$C = 20$	1 complex pair	$ADR = 30^\circ (w_n = 4)$
	1 real root	$(\alpha \cong 1.5)$
$20 \lesssim C \lesssim 30$	1 complex pair	$30^\circ < ADR < 45^\circ$
	1 real root	
$30 \lesssim C \lesssim 40$	1 complex pair	$45^\circ < ADR < 60^\circ$
	1 real root	
$40 \lesssim C \lesssim 250$	1 complex pair	$60^\circ < ADR < 90^\circ$
	1 real root	
$250 < C$	1 complex pair	$ADR > 90^\circ$
	1 real root	
	Unstable since one complex pair occurs in right half plane	

Fig. 21 Polynomial Contours from Photographs 3-3 Constant R and Constant θ Curves.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

CONCLUSIONS:

In this thesis, the mapping theorem was used in estimating the dynamic characteristics of physical systems from the characteristic equation of the performance function. Modification of a Fourier Harmonic Synthesizer greatly facilitated the application of the theorem, and enabled the polynomial contour to be obtained directly. The presentation of the polynomial contour on a standard cathode ray oscilloscope made it possible to apply the encirclement criteria conveniently and with considerable accuracy.

It was found that input signal levels too low to be indicated by the Sanborn Recorder pens still produced accurate polynomial contours on the scope presentation.

The difference between the computed polynomial contour check points and the corresponding points on the photo records was not measurable.

By expanding the scope presentation, any portion of the polynomial contour may be closely examined. This procedure is useful when investigating critical, or limiting values of the characteristic equation coefficients.

Examples have been included which indicate that the techniques presented are applicable either to polynomials with fixed coefficients or to polynomials whose coefficients are adjustable. In the case of adjustable coefficients, the method presented provides for the possibility of setting the limits on the coefficients which will allow the roots of the system characteristic equation to remain within a specified sector on the root complex plane.

This will allow the dynamic characteristics of the system to meet the prescribed specifications. The performance functions of the system components are also determined and are represented by by the coefficients. This knowledge would be very helpful to the preliminary designer.

Although the examples presented here are primarily chosen from servo problems, the technique is equally applicable to aircraft stability problems or to other problems where the dynamics determined by the characteristic equation are of interest.

RECOMMENDATIONS:

For the purpose of illustrating the analysis techniques for the adjustable coefficient case, a typical problem involving an equation of the third degree was chosen (Example 5). It is recommended that these procedures be applied to actual physical problems of fourth degree or higher. The results of this investigation indicate that the techniques developed apply equally well to equations of higher order without excessively increasing the amount of computation required.

It is further recommended that the feasibility of inverting this technique be studied. The shape of the polynomial contour is readily adjustable by the synthesizer potentiometers while the machine is in operation. Once the relation between the polynomial plane contour and the root plane is fully appreciated, it should be possible to adjust the synthesizer to produce polynomial contours representing the desired dynamic characteristics. By starting with the potentiometer settings, and inverting the scaling technique, the equation coefficients providing the specified dynamic characteristics may be determined.

If such a technique can be developed, it should facilitate further the problem of estimation of the dynamic characteristics of physical systems.

APPENDIX A

IMPLEMENTATION OF THE HARMONIC SYNTHESIZER MODIFICATIONS

The harmonic synthesizer used in obtaining data for this thesis was constructed by the Instrumentation Laboratory of the Massachusetts Institute of Technology. The complete circuit diagrams of the synthesizer are available at the Laboratory.[5] The mechanics of the synthesizer operation will be outlined in order to clarify the reasons for the modifications made.

This synthesizer is composed of the fundamental and 23 harmonic components. Figure A - 1 illustrates the operation of the fundamental component of the synthesizer. An electric motor, through a gear train, furnishes the drive for the rotation of the rotor coils, which are 90° apart electrically. The moving coils are excited by a 400 ~ voltage in the stator coils. The signal voltages are picked off at the summing resisters, amplified, and sent to a Sanborn two-channel recorder. Since the rotors for each component are 90° apart electrically, the resulting signal voltages will have a phase difference of 90° and thus correspond to $\cos\theta$ and $\sin\theta$, respectively, as the motor turns the rotor shaft through one complete revolution of 360°. Multiplication by R is done by the potentiometers. Thus, the output voltages of the fundamental component in the synthesizer correspond to $R_1 \cos\theta$ and $R_1 \sin\theta$, or to the real and imaginary terms, respectively, of

$$z_1 = R_1 e^{j\theta} = R_1 \cos\theta + j R_1 \sin\theta$$

Similarly, if the rotor coils of the 2^{nd} harmonic component have twice the angular velocity of the fundamental, the output voltages of this component correspond to

$$z_2 = R_2 e^{j 2\theta} = R_2 \cos 2\theta + j R_2 \sin 2\theta$$

$$\text{and } z_3 = R_3 e^{j 3\theta} = R_3 \cos 3\theta + j R_3 \sin 3\theta, \text{ etc.}$$

If $R_1 = BR$, $R_2 = AR^2$ and $R_3 = R^3$, then the sum of the first three components of the synthesizer is

$$P(R, \theta) = (R^3 \cos 3\theta + AR^2 \cos 2\theta + BR \cos \theta)$$
$$+ j (R^3 \sin 3\theta + AR^2 \sin 2\theta + BR \sin \theta).$$

Thus, the real and imaginary parts of any complex harmonic function (within the limits of the machine) may be readily obtained by expressing it as a polynominal in polar form, such as

$$P(R, \theta) - a_o = R^n e^{j n\theta} + a_{n-1} R^{n-1} e^{j(n-1)\theta} + \dots .$$
$$\dots . + a_2 R^2 e^{j2\theta} + a_1 R e^{j\theta}$$

The characteristic equation of any physical system may be expressed in this manner.

The real and imaginary parts of the function - henceforth referred to as the x and y signals, respectively - were applied to the x and y inputs of a standard Dumont cathode ray oscilloscope which was fitted with a Polaroid Land Camera

A z - dot generator was designed in order to provide scope illumination at the moments when the 400 ~ excitation signal was at its maximum, as shown in Figure A - 2.

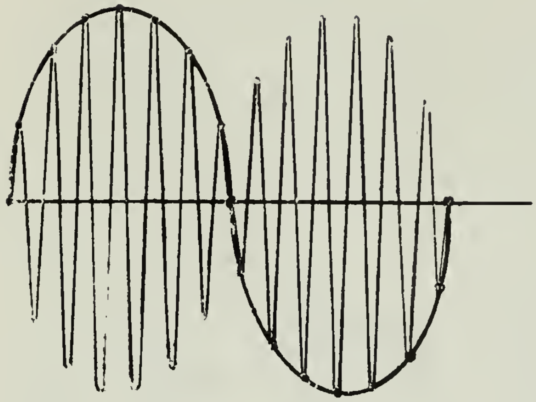


Figure A-2: Signal Voltage with Z-Dot Generation

These pulses appear on the face of the scope at the rate of 400/ second and appear to be a a steady dot, which traces the complex-vector sum of the x and y signal voltages. Thus, a visual representation of the polynominal contour is presented, which is the first requirement for the application of the encirclement criterion.

Finally, a cam-actuated microswitch was installed on the rotor shaft of the fundamental component, which provided timing indications corresponding to certain values of θ.. The final 180° of the dot travel, being symmetrical to the first 180°, was blanked out. Figure A - 3 shows the Fourier Harmonic Synthesizer after modification, and Figures A-4, 5, and 6 are the circuit diagrams of the x and y signal amplifiers and the z-dot generator. Figure 7 is a sketch of the cam.

In operation, as the dot traced out the plot of the characteristic equation polynomial , the microswitch cam follower opened contacts, thus blanking the scope, during the indicated intervals. The scope camera, set for time exposure, photographed the trace for a permanent record.

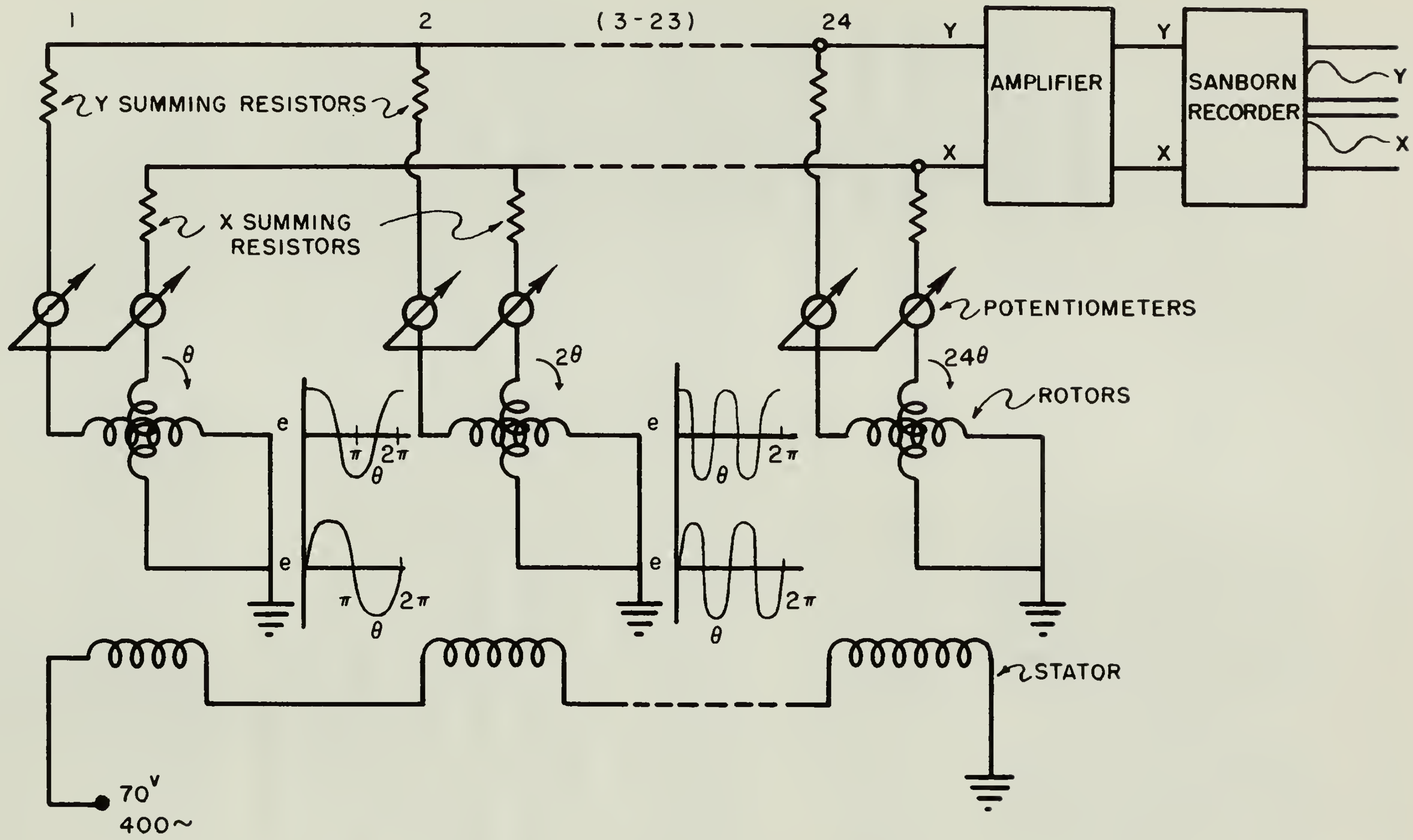
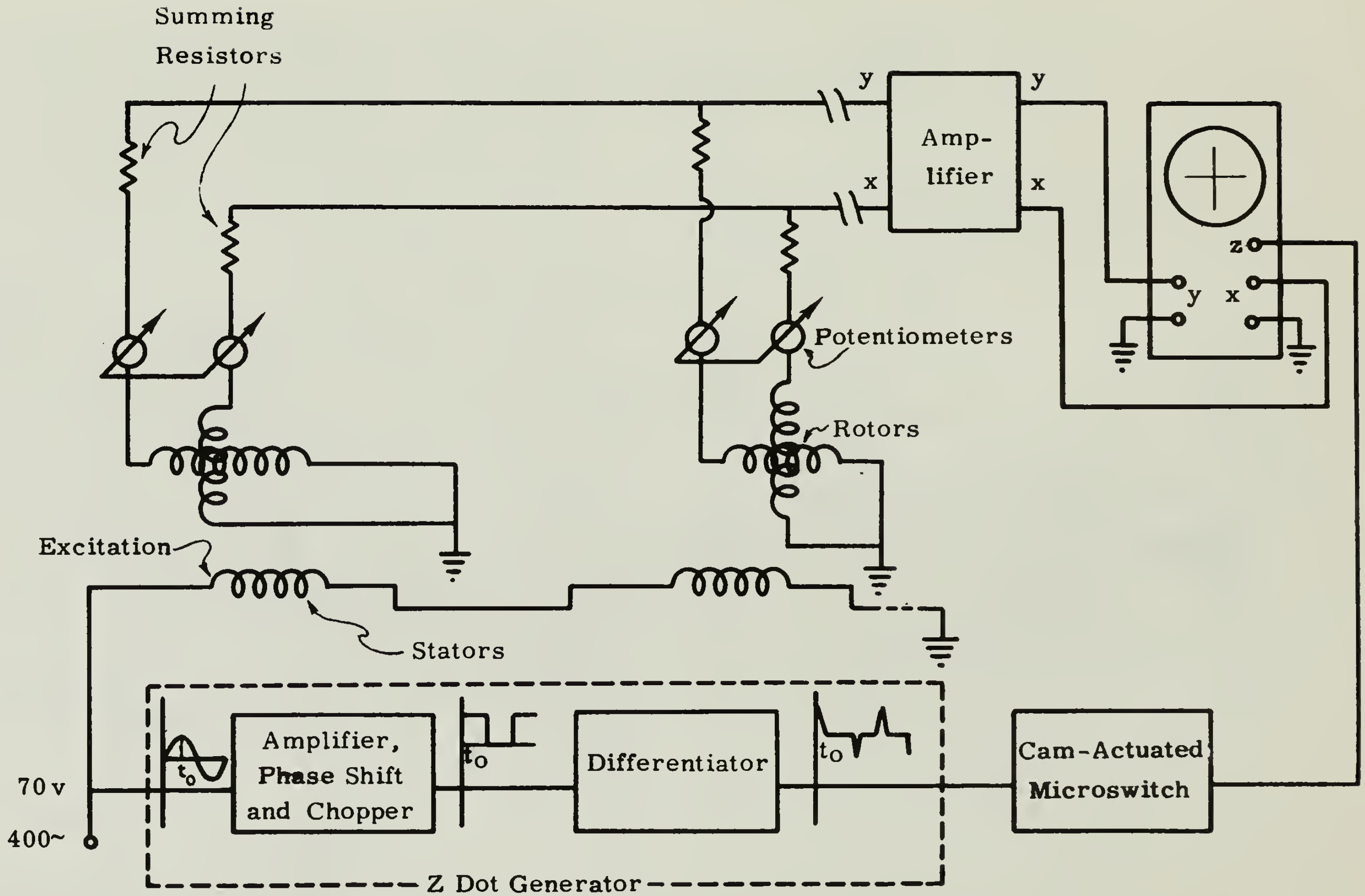
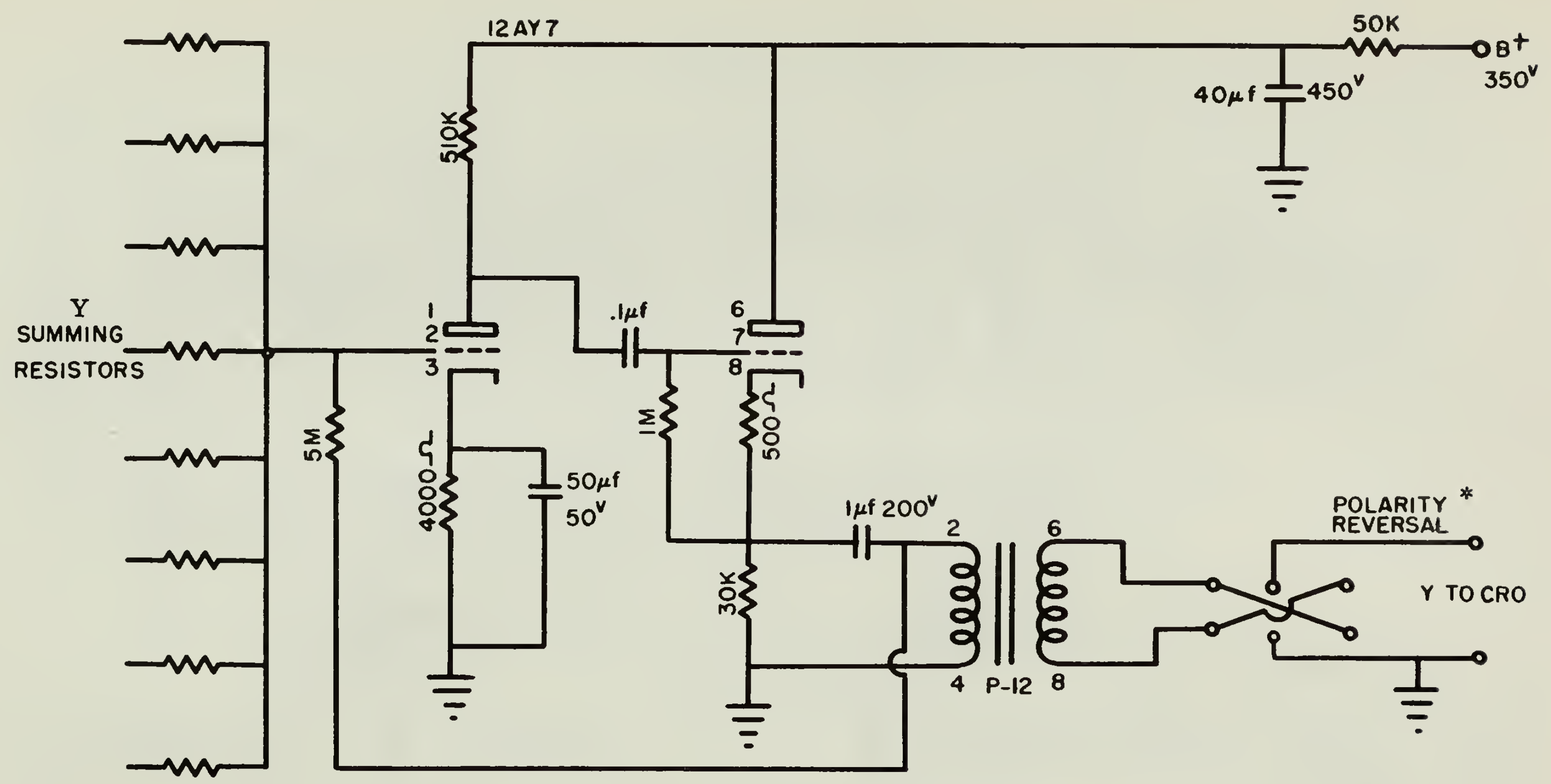


Figure A-1: Fourier Harmonic Synthesizer



Modified Fourier Harmonic Synthesizer

Figure A-3



* Polarity reversal was provided in order to insure control of the direction of the positive - going trace.

Figure A-4: Y Channel Amplifier

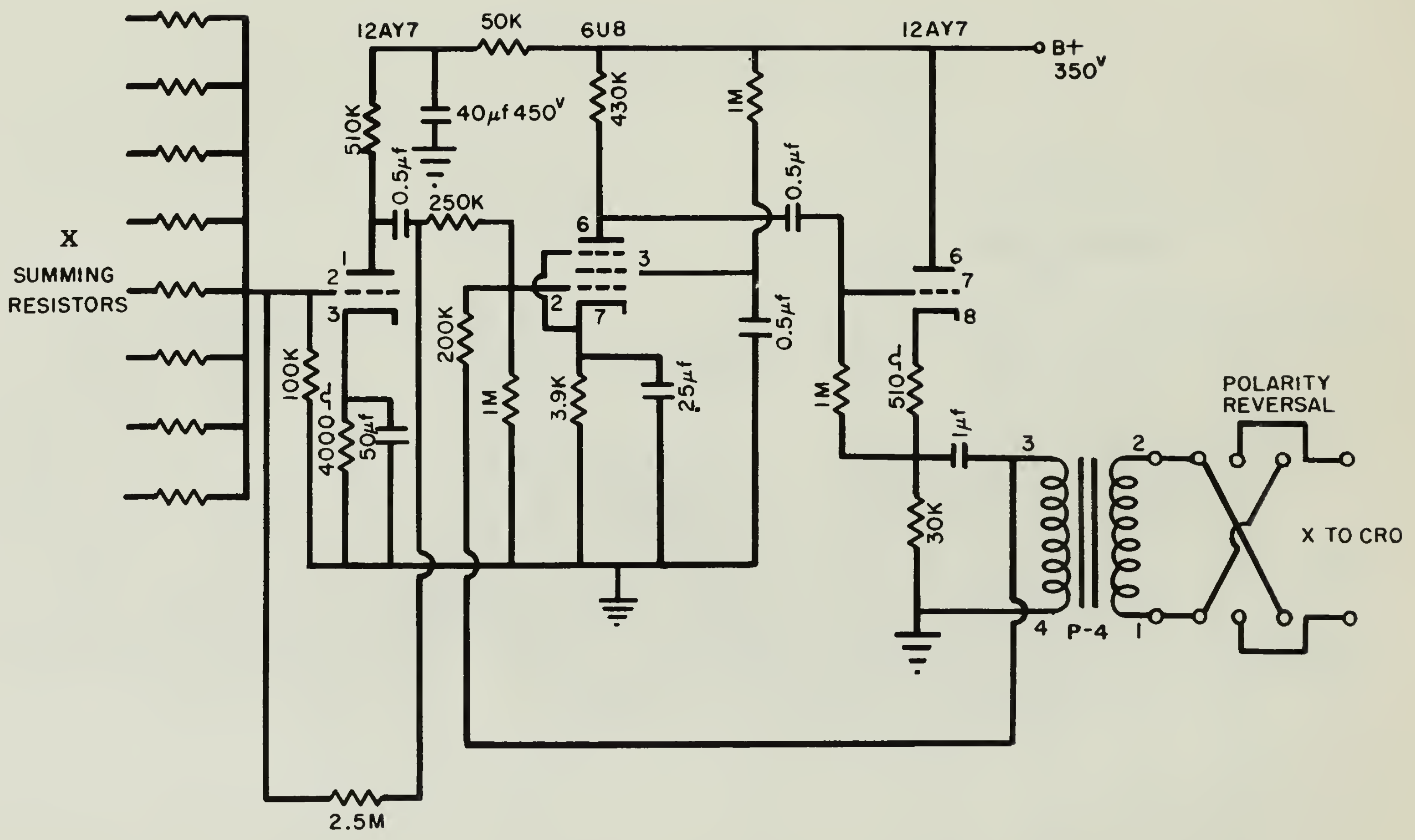
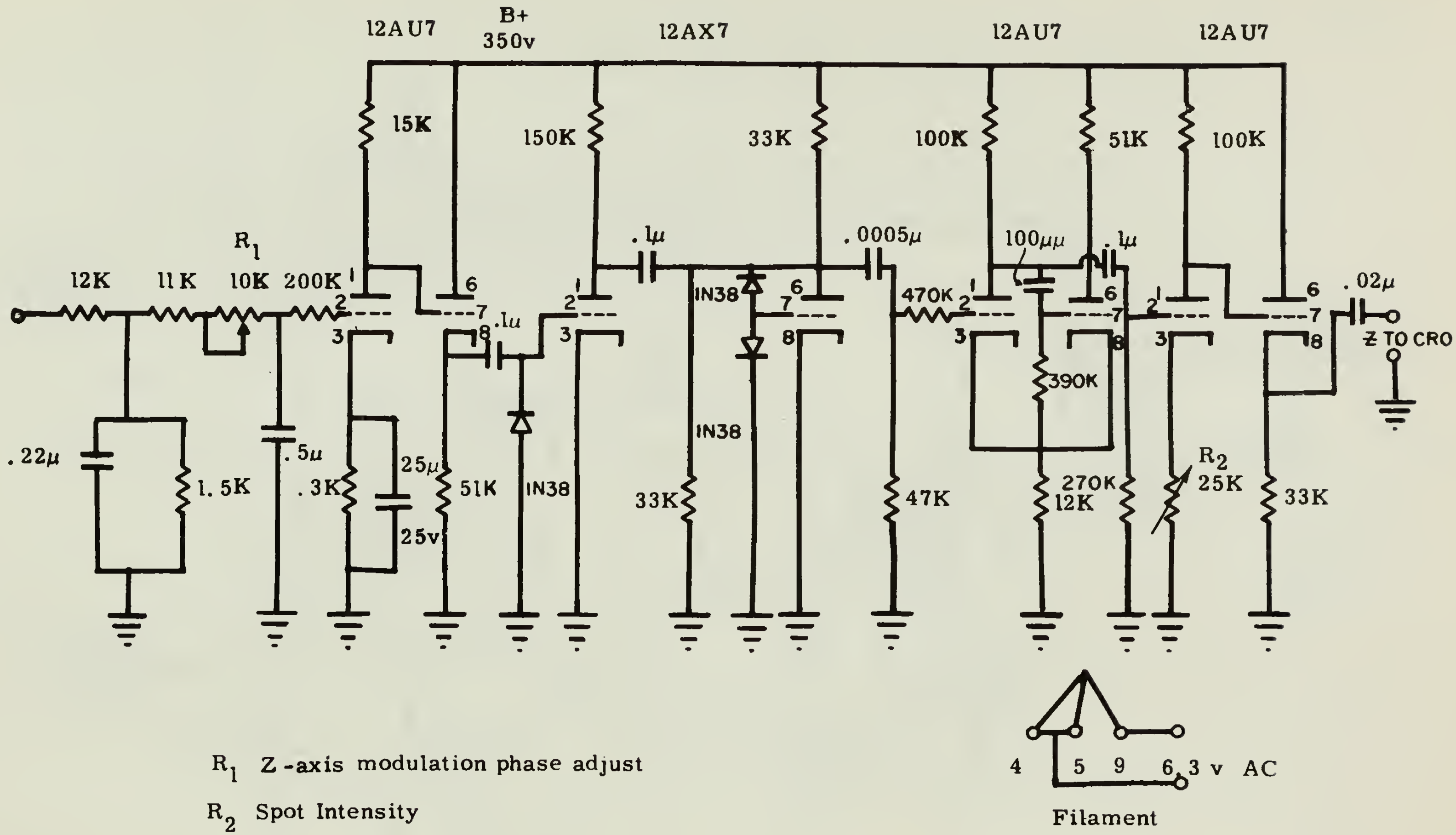


Figure A-5: X Channel Amplifier



R_1 Z-axis modulation phase adjust

R_2 Spot Intensity

Figure A-6: Z Dot Generator

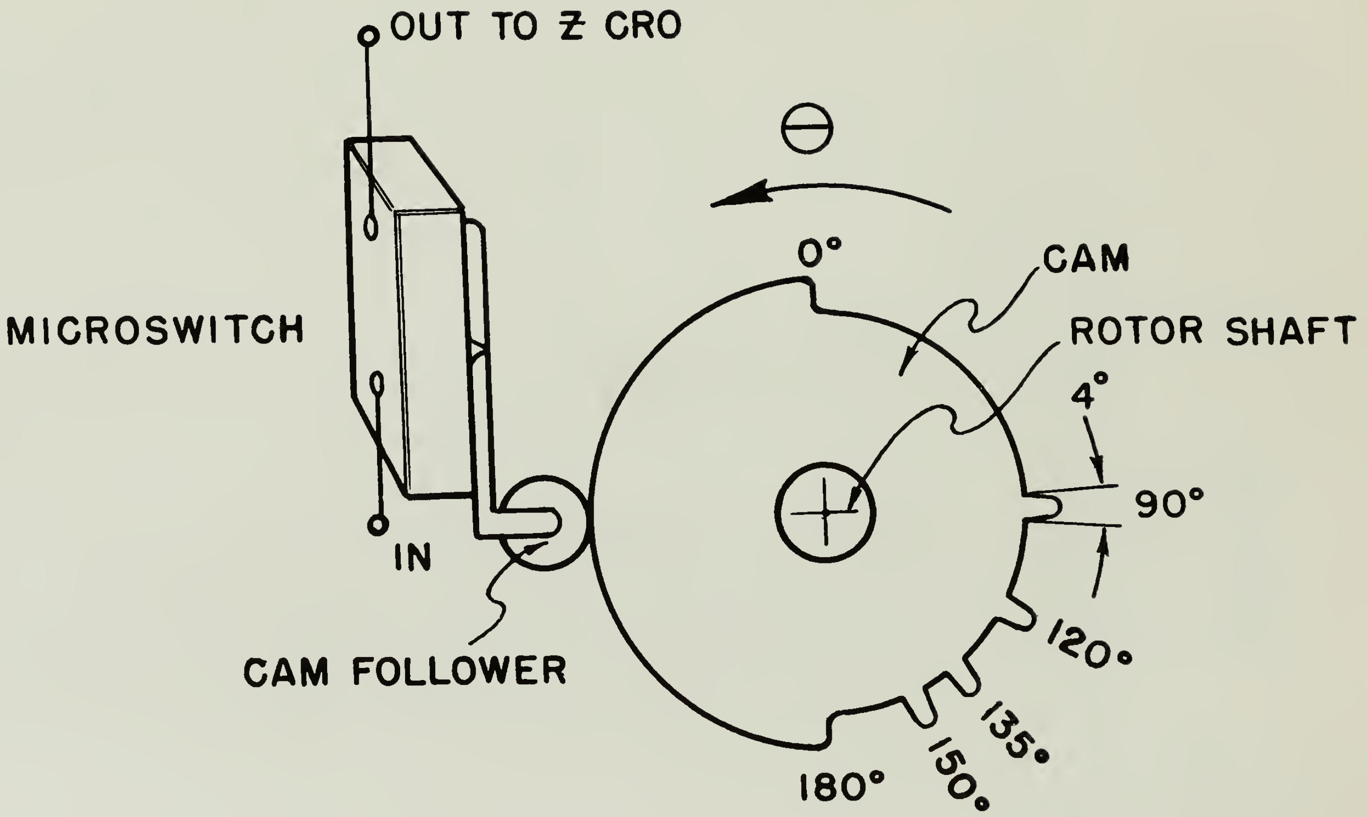


Figure A - 7: Cam - Actuated Microswitch

APPENDIX B

Tabulation of Numerical Values Used in the Determination of the Effect of Varying the Coefficients in the Characteristic Equation of Example 4:

$$P(z) = z^3 + (a+b)\, z^2 + z + b$$

$$= z^3 + (\alpha + 2\zeta_r \omega_r)\, z^2 + (2\zeta_r \omega_r \alpha + \omega_r^2)\, z + \alpha \omega_r^2 = 0$$

where $\quad a+b = \alpha + 2\zeta_r \omega_r$

$$1 = 2\zeta_r \omega_r \alpha + \omega_r^2$$

$$b = \alpha \omega_r^2$$

TABLE B-1

Values of $\zeta_r \ \omega_r$, b, and a+b when $\omega_r = 1$:

Using the relating equations between the roots and the coefficients, and setting $\omega_r = 1$, we have

$$\alpha = (a+b) - 2\,\zeta_r\,(1)\ , \qquad 2\,\zeta_r\,(1)\ \Big[(a+b) - 2\,\zeta_r\,(1)\Big] = 0$$

$$2\,\zeta_r = a+b,\ 0$$

$$\left\{\begin{array}{r} \zeta_r = \zeta_r\ \omega_r = \dfrac{a+b}{2}\ ,\ 0 \\ \alpha = 0,\ a+b \end{array}\right\}$$

a + b	α	$\zeta_r \ \omega_r$	b
1	1.0	0, .5	1, 0
1.1	1.1, 0	0, .55	1.1, 0
1.2	1.2, 0	0, .60	1.2, 0
1.3	1.3, 0	0, .65	1.3, 0
1.4	1.4, 0	0, .70	1.4, 0
1.6	1.6, 0	0, .80	1.6, 0
1.7	1.7, 0	0, .85	1.7, 0
1.732	1.732, 0	0, .866	1.732, 0
1.8	1.8, 0	0, .90	1.8, 0
1.9	1.9, 0	0, .95	1.9, 0
2.0	2.0, 0	0, 1.0	20, 0

TABLE B-2

By the same procedures as outlined above, ω_r is set at 0.9 and the equations obtained are:

$$\alpha = (a+b) - 1.8\,\zeta_r \quad ; \quad \zeta_r = \frac{a+b \pm \sqrt{(a+b)^2 - .76}}{2\,(1.8)} = \frac{N}{3.6}$$

$\omega_r = .9$

a+b	$(a+b)^2$	$(a+b)^2 - .76$	$\sqrt{(a+b)^2 - .76}$	N	ζ_r	$\zeta_r\,\omega_r$	$1.8\ \zeta_r$	α	b
1.0	1.000	.24	.491	1.491, .509	.415, .1413	.374, .1273	.748, .2546	.254, .746	.206, .605
1.1	1.21	.45	.672	1.772, .428	.492, .1189	.443, .107	.886, .214	.214, .886	.1735, .718
1.2	1.44	.68	.826	2.026, .374	.563, .1039	.506, .0935	1.012, .187	.187, 1.013	.1515, .820
1.3	1.69	.93	.965	2.265, .335	.629, .093	.566, .0837	1.132, .1674	.167, 1.133	.1353, .918
1.4	1.96	1.20	1.096	2.496, .304	.693, .0845	.624, .076	1.248, .152	.152, 1.248	.123, 1.01
1.5	2.25	1.49	1.222	2.722, .278	.756, .0772	.681, .0695	1.362, .139	.139, 1.361	.1126, 1.103
1.6	2.56	1.80	1.342	2.942, .258	.817, .0689	.735, .062	1.470, .124	.124, 1.476	.1005, 1.196
1.7	2.89	2.13	1.460	3.160, .240	.877, .0667	.789, .060	1.578, .120	.120, 1.580	.0972, 1.28
1.732	3.00	2.24	1.498	3.230, .234	.897, .065	.808, .0585	1.616, .1170	.116, 1.615	.094, 1.31
1.8	3.24	2.48	1.575	3.375, .225	.937, .0625	.843, .0562	1.686, .1124	.113, 1.687	.0915, 1.367
1.9	3.61	2.85	1.690	3.590, .210	.997, .0583	.897, .0525	1.794, .105	.105, 1.795	.085, 1.454
2.0	4.00	3.24	1.800	3.800, .200	1.055, .0555	.950, .050	1.90, .10	.100, 1.900	.081, 1.54
2.1	4.41	3.65	1.911	4.011, .189	1.113, .0525	1.000, .0472	2.000, .0944	.095, 2.005	.077, 1.624
2.2	4.84	4.08	2.020	4.220, .180	1.172, .0500	1.055, .045	2.110, .090	.090, 2.110	.073, 1.71
2.3	5.29	4.53	2.130	4.430, .170	1.230, .0472	1.107, .0425	2.214, .085	.085, 2.215	.069, 1.795
2.4	5.76	5.00	2.238	4.638, .162	1.287, .0450	1.158, .0405	2.316, .081	.081, 2.319	.0656, 1.88
2.5	6.25	5.49	2.346	4.846, .154	1.344, .0428	1.210, .0385	2.420, .077	.077, 2.423	.0624, 1.963

TABLE B - 3

$\omega_r = .8; \quad \alpha = (a+b) - 1.6\,\zeta_r \quad ; \quad \zeta_r = \dfrac{a+b \pm \sqrt{(a+b)^2 - 1.44}}{2(1.6)} = \dfrac{N}{3.2}$

a+b	$(a+b)^2$	$(a+b)^2 - 1.44$	$\sqrt{(a+b)^2 - 1.44}$	N	ζ_r	$\zeta_r\,\omega_r$	$1.6\,\zeta_r$	α	b
1	1	-.44							.128
1.1	1.21	-.23							
1.2	1.44	0	0	1.2, 1.2	.375	.300	.600	.6	.384
1.3	1.69	.25	.500	1.8, .8	.562, .250	.450, .200	.900, .400	.4, .9	.256, .576
1.4	1.96	.52	.722	2.122, 678	.663, .212	.530, .1695	1.061, .339	.339, 1.061	.217, .679
1.5	2.25	0.81	.900	2.4, .6	.750, .1875	.600, .150	1.200, .300	.300, 1.200	.192, .768
1.6	2.56	1.12	1.058	2.658, .542	.830, .1695	.664, .1356	1.328, .2712	.272, 1.328	.174, .850
1.7	2.89	1.45	1.205	2.905, .495	.908, .155	.727, .124	1.454, .248	.247, 1.453	.158, .930
1.8	3.24	1.80	1.343	3.143, .451	.983, .143	.787, .1144	1.574, .2288	.228, 1.572	.146, 1.006
1.9	3.61	2.17	1.475	3.375, .425	1.054, .133	.844, .1064	1.688, .2128	.213, 1.687	.1365, 1.08
2.0	4.00	2.56	1.600	3.600, .400	1.125, .125	.900, .1000	1.800, .2000	.200, 1.800	.128, 1.153
2.1	4.41	2.97	1.725	3.825, .375	1.195, .117	.956, .0937	1.912, .1874	.187, 1.913	.1197, 1.225
2.2	4.84	3.40	1.845	4.045, .355	1.263, .111	1.010, .0888	2.020, .1776	.179, 2.021	.1146, 1.294
2.3	5.29	3.85	1.963	4.263, .337	1.333, .1053	1.066, .0843	2.132, .1686	.169, 2.131	.1082, 1.364
2.4	5.76	4.32	2.08	4.48, .320	1.40, .100	1.120, .0800	2.240, .1600	.160, 2.240	.1025, 1.435
2.5	6.25	4.81	2.193	4.693, .307	1.467, .096	1.173, .0768	2.346, .1536	.154, 2.346	.0985, 1.500

TABLE B-4

$\omega_r = .7; \quad \alpha = (a+b) - 1.4\,\zeta_r \quad ; \zeta_r = \dfrac{a+b \pm \sqrt{(a+b)^2 - 2.04}}{2(1.4)} = \dfrac{N}{2.8}$

$a+b$	$(a+b)^2$	$(a+b)^2 - 2.04$	$\sqrt{(a+b)^2 - 2.04}$	N	ζ_r	$\zeta_r\,\omega_r$	$1.4\,\zeta_r$	α	b
1.4	1.96	-.08							
1.5	2.25	.21	.458	1.958, 1.042	.698, .372	.488, .260	.977, .521	.522, 978	.256, .479
1.6	2.56	.52	.722	2.322, .878	.828, .313	.579, .219	1.16, .438	.44, 1.16	.216, .568
1.7	2.89	.85	.923	2.623, .777	.937, .278	.655, .1945	1.312, .389	.389, 1.311	.191, .642
1.8	3.24	1.20	1.096	2.896, .704	1.032, .251	.723, .176	1.444, .351	.35, 1.45	.172, .710
1.9	3.61	1.57	1.254	3.154, .646	1.125, .231	.787, .162	1.575, .323	.324, 1.576	.159, .772
2.0	4.00	1.96	1.400	3.400, .600	1.214, .214	.850, .150	1.700, .299	.299, 1.701	.1465, .834
2.1	4.41	2.37	1.540	3.640, .560	1.300, .200	.910, .140	1.820, .280	.28, 1.82	.137, .892
2.2	4.84	2.80	1.675	3.875, .525	1.383, .1875	.969, .1312	1.935, .262	.263, 1.937	.129, .949
2.3	5.29	3.25	1.805	4.105, .495	1.467, .177	1.026, .124	2.052, .248	.248, 2.052	.1216, 1.005
2.4	5.76	3.72	1.930	4.33, .470	1.545, .168	1.082, .1176	2.164, .235	.236, 2.164	.1157, 1.061
2.5	6.25	4.21	2.05	4.55, .450	1.624, .1607	1.137, .1124	2.274, .2248	.226, .2.274	.1108, 1.115

TABLE B - 5

$$\omega_r = .6 \;;\quad \alpha = (a+b) - 1.2\,\zeta_r \;;\quad \zeta_r = \frac{a+b \pm \sqrt{(a+b)^2 - 2.56}}{2(1.2)} = \frac{N}{2.4}$$

(a+b)	$(a+b)^2$	$(a+b)^2 - 2.56$	$\sqrt{(a+b)^2 - 2.56}$	N	ζ_r	$\zeta_r \omega_r$	$1.2\,\zeta_r$	α	b
1.6	2.56	0	0	1.60	.667	.400	.800	.800	.288
1.7	2.89	.33	.575	2.275, 1.125	.948, .468	.569, .281	1.138, .562	.562, 1.138	.2025, .409
1.732	3.00	.44	.664	2.396, 1.068	.998, .445	.599, .267	1.198, .534	.534, 1.198	.192 , .431
1.8	3.24	.68	.826	2.626, .974	1.093, .405	.656, .243	1.314, .486	.486, 1.314	.175, .473
1.9	3.61	1.05	1.025	2.925, .875	1.218, .364	.730, .2185	1.462, .437	.437, 1.462	.1575, .527
2.0	4.00	1.44	1.20	3.2 , .8	1.333, .333	.800, .200	1.60, .400	.400, 1.600	.144, .576
2.1	4.41	1.85	1.36	3.46, .74	1.441, .308	.865, .185	1.73, .370	.370, 1.730	.1333, .623
2.2	4.84	2.28	1.51	3.71, .69	1.546, .287	.927, .1722	1.856, .344	.344, 1.856	.124, .668
2.3	5.29	2.73	1.654	3.954, .646	1.648, .269	.988, .1615	1.977, .323	.323, 1.977	.1163, .712
2.4	5.76	3.20	1.79	4.19, .61	1.745, .254	1.046, .1525	2.095, .305	.305, 2.095	.1097, .754
2.5	6.25	3.69	1.924	4.424, .576	1.845, .240	1.108, .144	2.215, .288	.287, 2.213	.1033, .797

TABLE B - 6

$$\omega_r = .5 \; ; \quad \alpha = (a+b) - \zeta_r \quad ; \quad \zeta_r = \frac{(a+b) \pm \sqrt{(a+b)^2 - 3}}{2(1)} = \frac{N}{2}$$

a+b	$(a+b)^2$	$(a+b)^2 - 3.00$	$\sqrt{(a+b)^2 - 3.00}$	N	ζ_r	$\zeta_r \omega_r$	α	b
1.7	2.89							
1.8	3.24	.24	.491	2.291, 1.309	1.145, .654	.572, .327	.654, 1.146	.1635, .286
1.9	3.61	.61	.782	2.682, 1.118	1.341, .559	.670, .2785	.559, 1.341	.140, .335
2.0	4.00	1.00	1.000	3.000, 1.000	1.500, .500	.750, .250	.500, 1.500	.125, .375
2.1	4.41	1.41	1.188	3.288, .912	1.644, .456	.822, .228	.456, 1.644	.1142, .411
2.2	4.84	1.84	1.357	3.557, .843	1.778, .422	.889, .211	.422, 1.778	.1056, .444
2.3	5.29	2.29	1.515	3.815, .785	1.907, .393	.953, .1965	.393, 1.907	.0982, .476
2.4	5.76	2.76	1.663	4.063, .737	2.031, .369	1.015, .1845	.369, 2.031	.0923, .507
2.5	6.25	3.25	1.805	4.305, .695	2.152, .348	1.076, .174	.348, 2.152	.0870, .537

TABLE B-7

$$\omega_r = .4 \quad ; \quad \alpha = (a+b) - .8\ \zeta_r \quad ; \quad \zeta_r = \frac{a+b \pm \sqrt{(a+b)^2 - 3.36}}{2(.8)} = \frac{N}{1.6}$$

$(a+b)$	$(a+b)^2$	$(a+b)^2 - 3.36$	$\sqrt{(a+b)^2 - 3.36}$	N	ζ_r	$\zeta_r \omega_r$	$.8\ \zeta_r$	α	b
1.9	3.61	.25	.500	2.400, 1.400	1.500, .875	.600, .350	1.200, .700	.700, 1.200	.112, .192
2.0	4.00	.64	.800	2.800, 1.200	1.750, .750	.700, .300	1.400, .600	.600, 1.400	.096, .224
2.1	4.41	1.05	1.025	3.125, 1.075	1.955, .672	.782, .269	1.564, .538	.537, 1.563	.086, .250
2.2	4.84	1.48	1.216	3.416, .984	2.130, .615	.853, .246	1.706, .492	.493, 1.707	.0789, .273
2.3	5.29	1.93	1.390	3.690, .910	2.310, .568	.925, .227	1.848, .452	.452, 1.848	.0724, .296
2.4	5.76	2.40	1.550	3.950, .850	2.470, .531	.988, .2125	1.976, .425	.425, 1.975	.068, .316
2.5	6.25	2.89	1.701	4.201, .799	2.630, .499	1.053, .1998	2.106, .399	.399, 2.101	.0639, .336

APPENDIX C

Tabulation of Fourier Harmonic Synthesizer Scale Settings Used in Example 5.

The basic equation used in this example is

$$P(z) = z^3 + (a+b)\, z^2 + (ab+1)\, z + \frac{b}{c}$$

or, with $z = Re^{j\theta}$

$$P(R,\theta) - \frac{b}{c} = R^3 e^{j3\theta} + (a+b)\, R^2 e^{j2\theta} + (ab+1)\, Re^{j\theta}$$

At $\theta = 0$, $e^{j\theta} = e^{j0} = 1 + j0$

$$e^{j2\theta} = e^{j0} = 1 + j0$$

$$e^{j3\theta} = e^{j0} = 1 + j0$$

and the real part of $P(R,\theta) - \frac{b}{c}\Big]_{\theta=0} = F_o = R^3 + (a+b)\, R^2 + (ab+1)\, R$

Similarly, at $\theta = \pi$, $e^{j\theta} = e^{j\pi} = -1 + j0$

$$e^{j2\theta} = e^{j2\pi} = 1 + j0$$

$$e^{j3\theta} = e^{j3\pi} = -1 + j0$$

and the real part of $P(R,\theta) - \frac{b}{c}\Big]_{\theta=\pi} = F_\pi = -R^3 + (a+b)\, R^2 - (ab+1)\, R$

Since, in this example, $R_{max} = 6$, the largest potentiometer setting, $S_{3_{max}}$, will correspond to R^3_{max} throughout. That is, division of all numerical coefficients by $R^3_{max} = 6^3 = 216$ gives the scale setting for the potentiometer in each run.

These tabulations are carried out in the following tables, as indicated.

Sample calculation:

$$R = 6 \quad R^2 = 36 \qquad R^3 = 216$$

$$1.3\,R^2 = 1.3\,(36) = 46.8$$

$$1.3\,R = 1.3\,(6) = 7.8$$

$$S_3 = \frac{1}{216} R^3 = \frac{1}{216} (216) = 1.000 \qquad F_o = 216 + 46.8 + 7.8 = 270.6$$

$$S_2 = \frac{1}{216} (1.3R^2) = \frac{1}{216} (46.8) = 0.217 \qquad F_\pi = -216 + 46.8 - 7.8 = -177$$

$$S_1 = \frac{1}{216} (1.3R) = \frac{1}{216} (7.8) = 0.0361$$

TABLE C - 1

Table	Photo No.	Scope Settings (Gain = 1) Y Amp.	X Amp.	Multiply S_1, S_2 and S_3 by
C-2	1-1a	29	61.5	1
C-2	1-1b	29	61.5	100
C-3	1-2'a	24	55	1
C-3	1-2'b	24	55	20
C-4	1-2 a	18	44	1
C-4	1-2 b	18	44	10
C-5	1-3 a	22	48	1
C-5	1-3b	22	48	5
C-6	2-1a	30	64	1
C-6	2-1b	30	64	10
C-7	2-2'a	24	55	1
C-7	2-2'b	21	39	10
C-8	2-2 a	29	61	1
C-8	2-2b	29	61	10
C-9	2-3 a	30	64	1
C-9	2-3 b	20	45	20
C-10	3-1a	20	37	1
C-10	3-1b	20	37	5
C-11	3-2 a	30	59	1
C-11	3-2 b	30	59	20
C-12	3-3a	30	61	1
C-12	3-3b	20	40	10

TABLE C - 2

$$P(z) - \frac{b}{c} = z^3 + (a+b)\, z^2 + (ab+1)\, z \quad ; \qquad a+b = 1.3\,, \quad ab+1 = 1.3$$

$$P(R,\theta) - \frac{b}{c} = R^3\, e^{j3\theta} + R^2\, 1.3\, e^{j2\theta} + R\, 1.3\, e^{j\theta}$$

R	R^2	R^3	$1.3\, R^2$	$1.3\, R$	S_3	S_2	S_1	F_o	F_π
6	36	216	46.8	7.8	1	.2170	.0361	270.6	-177.0
4	16	64	20.8	5.2	.2960	.0962	.02405	90.0	-48.4
3	9	27	11.7	3.9	.1250	.0541	.01805	42.6	-19.2
2	4	8	5.2	2.6	.0370	.02403	.0120	15.8	- 5.4
1	1	1	1.3	1.3	.00463	.00602	.00602	3.6	- 1.0
.5	.25	.125	.325	.65	.000579	.001505	.00301	1.1	- .45
.2	.04	.008	.052	.26	.000037	.00024	.0012	.320	- .216

TABLE C - 3

$$P(z) - \frac{b}{c} = z^3 + (a+b)\ z^2 + (ab+1)\ z \ ; \quad a + b = 1.3 \quad , \quad ab+1 = 1.42$$

$$P(R, \theta) - \frac{b}{c} = R^3\ e^{j3\theta} + 1.3\ R^2\ e^{j2\theta} + 1.42\ R\ e^{j\theta}$$

R	R^2	R^3	$1.3R^2$	1.42 R	S_3	S_2	S_1	F_o	$F\pi$
5	25	125	32.5	7.10	.578	.1505	.0328	164.6	-99.6
4	16	64	20.8	5.68	.2960	.0962	.0263	90.5	-48.9
3	9	27	11.7	4.26	.1250	.0541	.0197	43.0	-19.56
2	4	8	5.2	2.84	.0370	.02403	.01315	16.04	- 5.64
1.5	2.25	3.38	2.92	2.13	.01565	.0136	.00986	8.43	- 2.59
1	1	1	1.3	1.42	.00463	.00602	.00656	3.72	- 1.12
.5	.25	.125	.325	.71	.000579	.001505	.00328	1.16	- .51

TABLE C - 4

$$P(z) - \frac{b}{c} = z^3 + (a+b)\,z^2 + (ab+1)\,z \;;\quad a+b = 1.3\,,\quad ab+1 = 10$$

$$P(R,\theta) - \frac{b}{c} = R^3 e^{j3\theta} + R^2\, 1.3\, e^{j2\theta} + R\, 10\, e^{j\theta}$$

R	R^2	R^3	$1.3R^2$	10 R	S_3	S_2	S_1	F_o	$F\pi$
6	36	216	46.8	60	1	.217	.278	322.8	-229.2
4	16	64	20.8	40	.296	.0962	.1852	124.8	-83.2
3	9	27	11.7	30	.125	.0541	.139	68.7	-45.3
2	4	8	5.2	20	.037	.02403	.0926	33.2	-22.8
1	1	1	1.3	10	.00463	.00602	.0463	12.3	-9.7
.5	.25	.125	.325	5	.000579	.001505	.0232	5.45	-4.8
.2	.04	.008	.052	2	.000037	.0002401	.00926	2.06	-1.956

TABLE C-5

$$P(z) - \frac{b}{c} = z^3 + (a+b)\, z^2 + (ab+1)\, z \; ; \quad a+b = 1.3 \;, \quad ab+1 = 19$$

$$P(R,\theta) - \frac{b}{c} = R^3 \, e^{j3\theta} + R^2 \, 1.3 \, e^{j2\theta} + R \, 19 \, e^{j\theta}$$

R	R^2	R^3	$1.3\,R^2$	19 R	S_3	S_2	S_1	F_o	F_π
5	25	125	32.5	95	.579	.1505	.440	252.5	-187.5
4	16	64	20.8	76	.296	.0962	.352	160.8	-119.2
3	9	27	11.7	57	.125	.0541	.264	95.7	- 72.3
2	4	8	5.2	38	.037	.02403	.176	51.2	- 40.8
1	1	1	1.3	19	.00463	.00602	.088	21.3	- 18.7
.5	.25	.125	.325	9.5	.000579	.001505	.044	9.95	- 9.3
.2	.04	.008	.052	3.8	.000037	.0002401	.0176	3.86	- 3.756

TABLE C-6

$$P(z) - \frac{b}{c} = z^3 + (a+b)\, z^2 + (ab+1)\, z \;;\quad a+b = 3.5, \quad ab+1 = 1.3$$

$$P(R,\theta) - \frac{b}{c} = R^3 e^{j3\theta} + R^2\, 3.5\, e^{j2\theta} + R\, 1.3\, e^{j\theta}$$

R	R^2	R^3	$3.5\,R^2$	$1.3R$	S_3	S_2	S_1	F_o	F_π
4	16	64	56	5.2	.296	.259	.02405	125.2	-13.2
3	9	27	31.5	3.9	.125	.146	.01805	62.4	+ 0.6
2	4	8	14	2.6	.037	.0648	.01205	24.6	+ 3.4
1	1	1	3.5	1.3	.00463	.0162	.00602	5.8	+ 1.2
.5	.25	.125	.875	.65	.000579	.00405	.00301	1.65	+.10
.2	.04	.008	.14	.26	.000037	.000648	.001205	.408	-.128

TABLE C-7

$$P(z) - \frac{b}{c} = z^3 + (a+b)\,z^2 + (ab+1)z; \quad a+b = 3.5\,, \quad ab+1 = 4.06$$

$$P(R,\theta) - \frac{b}{c} = R^3 e^{j3\theta} + 3.5\,R^2 e^{j2\theta} + 4.06\,R\,e^{j\theta}$$

R	R^2	R^3	$3.5\,R^2$	$4.06\,R$	S_3	S_2	S_1	F_o	F_π
5	25	125	87.5	20.3	.578	.405	.094	232.8	-57.8
4	16	64	56	16.25	.2960	.259	.0752	136.25	-24.25
3	9	27	31.5	12.2	.1250	.146	.0565	70.7	- 7.7
2	4	8	14	8.12	.0370	.0648	.0376	30.12	- 2.12
1.5	2.25	3.38	7.87	6.09	.01565	.0364	.0282	17.34	- 1.60
1	1	1	3.5	4.06	.00463	.0162	.0188	8.56	- 1.56
.5	.25	.125	.875	2.03	.000579	.00405	.0094	3.03	- 1.28
.2	.04	.008	.140	.812	.000037	.000648	.00376	.96	- .68

TABLE C-8

$$P(z) - \frac{b}{c} = z^3 + (a+b)\ z^2\ (ab+1)\ z\ ; \quad a+b = 3.5, \quad ab+1 = 10$$

$$P(R,\theta) - \frac{b}{c} = R^3\ e^{j3\theta} + 3.5\ R^2\ e^{j2\theta} + 10\ R\ e^{j\theta}$$

R	R^2	R^3	$3.5\ R^2$	10 R	S_3	S_2	S_1	F_o	F_π
6	36	216	126	60	1.0	.5830	.2780	402	-150
4	16	64	56	40	.2960	.2590	.1852	160	-48
3	9	27	31.5	30	.125	.146	.139	88.5	-25.5
2	4	8	14	20	.0370	.0648	.0926	42	-14
1	1	1	3.5	10	.0046	.0162	.0463	14.5	- 7.5
.5	.25	.125	.875	5	.0006	.0041	.0231	6	- 4.25
.2	.04	.008	.140	2	.000037	.0006	.0093	2.148	-1.868

TABLE C-9

$$P(z) - \frac{b}{c} = z^3 + (a+b)\,z^2 + (ab+1)\,z \,; \quad a+b = 3.5\,, \quad ab+1 = 19$$

$$P(R,\theta) - \frac{b}{c} = R^3\, e^{j3\theta} + 3.5R^2\, e^{j2\theta} + 19\,R\, e^{j\theta}$$

R	R^2	R^3	$3.5R^2$	19 R	S_3	S_2	S_1	F_o	F_π
5	25	125	87.5	95	.579	.405	.440	307.5	-132.5
4	16	64	56	76	.2960	.2590	.3520	196	-84
3	9	27	31.5	57	.125	.146	.264	115.5	-52.5
2	4	8	14	38	.0370	.0648	.1760	60	-32
.5	.25	.125	.875	9.5	.000579	.00405	.0440	10.5	-8.75
.2	.04	.008	.140	3.8	.000037	.000648	.0176	3.948	-3.668
.1	.01	.001	.035	1.9	.00000462	.000162	.0088	1.936	-1.876

TABLE C - 10

$$P(z) - \frac{b}{c} = z^3 + (a+b)\ z^2 + (ab+1)\ z\ ; \quad a+b = 6\ , \quad ab+1 = 1.3$$

$$P(R, \theta) - \frac{b}{c} = R^3\ e^{j3\theta} + 6\ R^2\ e^{j2\theta} + 19\ R\ e^{j\theta}$$

R	R^2	R^3	$6\ R^2$	1.3 R	S_3	S_2	S_1	F_o	F_π
6	36	216	216	7.8	1.0	1.0	.0361	439.8	-7.8
5	25	125	150	6.5	.579	.695	.0301	281.5	18.5
4	16	64	96	5.2	.2960	.4445	.0241	165.2	26.8
2	4	8	24	2.6	.0370	.1111	.0120	34.6	13.4
1	1	1	6	1.3	.00463	.0278	.0060	8.3	3.7
.5	.25	.125	1.5	.65	.000579	.00695	.0030	2.275	.725

<u>TABLE C - 11</u>

$$P(z) - \frac{b}{c} = z^3 + (a+b)z^2 + (ab+1)z \; ; \; a+b = 6 \; , \quad ab+1 = 10$$

$$P(R,\theta) - \frac{b}{c} = R^3 e^{j3\theta} + 6R^2 e^{j2\theta} + 10 R e^{j\theta}$$

R	R^2	R^3	$6R^2$	10 R	S_3	S_2	S_1	F_o	F_π
6	36	<u>216</u>	<u>216</u>	60	1.0	1.0	.2780	492	-60
5	25	125	150	50	.579	.695	.231	325	-25
4	16	64	96	40	.2960	.4445	.1852	200	- 8
2	4	8	24	20	.0370	.1111	.0926	52	- 4
1	1	1	6	10	.00463	.0278	.0463	17	- 5
.5	.25	.125	1.5	5	.0006	.00695	.0231	6.625	- 3.625
.1	.01	.001	.060	1	.00000463	.000278	.00463	1.061	-.941

TABLE C-12

$$P(z) - \frac{b}{c} = z^3 + (a+b) z^2 + (ab+1) z \quad ; \quad a+b = 6 , \qquad ab+1 = 19$$

$$P(R, \theta) - \frac{b}{c} = R^3 e^{j3\theta} + 6 R^2 e^{j2\theta} + 19 R e^{j\theta}$$

R	R^2	R^3	$6 R^2$	19 R	S_3	S_2	S_1	F_o	F_π
6	36	216	216	114	1.0	1.0	.5275	546	-114
5	25	125	150	95	.579	.695	.440	370	- 70
4	16	64	96	76	.2960	.4445	.3520	236	- 44
2	4	8	24	38	.0370	.1111	.1760	70	- 22
1	1	1	6	19	.00463	.0278	.0880	26	- 14
.5	.25	.125	1.5	9.5	.000579	.00695	.0440	11.125	- 8.125
.2	.04	.008	.24	3.8	.000037	.0011	.0176	4.048	- 3.568
.1	.01	.001	.06	1.9	.00000463	.000278	.0088	1.961	- 1.841

APPENDIX D

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